

# Stability of Stationary Sets in Nonlinear Systems with Set-valued Friction

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**Abstract—** In this paper we present conditions under which an equilibrium set of a multi-degree-of-freedom *nonlinear* mechanical system, with set-valued friction and an arbitrary number of frictional bilateral constraints, is attractive. These systems form an important class of hybrid engineering systems. The attractivity results are obtained using the framework of differential inclusions together with a Lyapunov-type stability analysis and LaSalle's invariance principle. The special structure of mechanical systems allows for a natural Lyapunov function candidate and a generic result for a large class of systems. Moreover, an instability theorem for assessing the instability of equilibrium sets of differential inclusions is presented. These results are illustrated by means of an example of a nonlinear mechanical system exhibiting both attractive and unstable equilibrium sets.

## I. INTRODUCTION

Most stability studies of discontinuous (hybrid) systems have focussed on isolated equilibrium points (or periodic orbits). However, in many systems discontinuities may induce the existence of entire connected sets of equilibria (called stationary sets or equilibrium sets), which can dramatically influence the dynamics. For a wide range of both controlled and uncontrolled (electro-mechanical) systems, dry friction can seriously affect the performance through the presence of such equilibrium sets. More specifically, the stiction phenomenon in friction can induce the presence of equilibrium sets, see for example [1]. The stability properties of such equilibrium sets is of major interest when analysing the global dynamic behaviour of these systems.

The aim of the paper is to present theoretical results which can be used to rigorously prove the conditional attractivity or instability of the equilibrium sets of nonlinear mechanical systems with frictional bilateral constraints.

The dynamics of mechanical systems with set-valued friction laws are described by differential inclusions of Filippov-type, see [2], [3] and references therein. Filippov systems, describing systems with friction, can exhibit equilibrium sets, which correspond to the stiction behaviour of those systems. Many publications deal with stability and attractivity properties of (sets of) equilibria in differential inclusions [4], [5], [6], [7], [8]. For example, in [4], [5] the attractivity of the equilibrium set of a dissipative one-degree-of-freedom friction oscillator with one switching boundary (i.e. one dry

friction element) is discussed. Moreover, in [5], [6], [7] the Lyapunov stability of an equilibrium point in the equilibrium set is shown. Most papers are limited to either one-degree-of-freedom systems or to systems exhibiting only one switching boundary. Very often, the stability properties of an equilibrium point in the equilibrium set is investigated and not the stability properties of the set itself. In this context, it is worth mentioning that in the more general scope of discontinuous systems, a range of results regarding stability conditions for *isolated* equilibria are available, see for example [9] in which conditions for stability are formulated in terms of the existence of common quadratic or piece-wise quadratic Lyapunov functions. In [10], extensions are given of the absolute stability problem and the Lagrange-Dirichlet theorem for systems with monotone multi-valued mappings (such as for example Coulomb friction). In the absolute stability framework, strict passivity properties of a linear part of the system are required for proving the asymptotic stability of an isolated equilibrium point, which may be rather restrictive for mechanical systems in general. Moreover, certain stability properties of equilibria in discontinuous systems of Lur'e type are studied in [11]. In a previous publication [12], we provided conditions under which the equilibrium set of a multi-degree-of-freedom *linear* mechanical system with an arbitrary number of Coulomb friction elements is attractive using Lyapunov-type stability analysis and LaSalle's invariance principle. Moreover, dissipative as well as non-dissipative linear systems have been considered.

In this paper we will give conditions under which an equilibrium set of a multi-degree-of-freedom *nonlinear* mechanical system with an arbitrary number of frictional bilateral constraints is attractive. The theorem for attractivity is proved using the framework of differential inclusions together with a Lyapunov-type stability analysis and LaSalle's invariance principle. The special structure of mechanical multi-body systems allows for a natural choice of the Lyapunov function and a systematic derivation of the proof for this large class of systems.

The modelling of nonlinear mechanical systems with set-valued dry friction laws by differential inclusions is discussed in Section II. Subsequently, the attractivity properties of an equilibrium set of a system with friction are studied in Section III-A. Moreover, the attractivity analysis provides an estimate for the region of attraction of the equilibrium set. In Section III-B, an instability theorem for differential inclusions is presented. These results are illustrated by an example in Section IV. Finally, concluding remarks are given in Section V.

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## II. MECHANICAL SYSTEMS WITH SET-VALUED FRICTION LAWS

We consider nonlinear mechanical systems with  $n$  degrees of freedom and  $m$  bilateral constraints with set-valued dry friction (frictional sliders). These kind of systems are very common in engineering practice (think for example of industrial robots, automotive drivelines and many others) and form an important class of hybrid engineering systems. We assume that a set of independent generalised coordinates  $\mathbf{q} \in \mathbb{R}^n$  is known, for which the  $m$  bilateral constraints are eliminated from the formulation of the dynamics of the system. We formulate the dynamics of the system using a Lagrangian approach, resulting in

$$\left( \frac{d}{dt} (T, \dot{\mathbf{q}}) - T, \mathbf{q} + U, \mathbf{q} \right)^T = \mathbf{f}^{\text{nc}}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{W}_T(\mathbf{q}) \boldsymbol{\lambda}_T, \quad (1)$$

or, alternatively,

$$\mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} - \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{W}_T(\mathbf{q}) \boldsymbol{\lambda}_T. \quad (2)$$

Herein,  $\mathbf{M}(\mathbf{q}) = \mathbf{M}^T(\mathbf{q}) > 0$  is the mass-matrix and  $T = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}}$  represents kinetic energy. Moreover,  $U$  denotes the potential energy. The column-vector  $\mathbf{f}^{\text{nc}}$  in (1) represents all smooth generalised non-conservative forces. The state-dependent column-vector  $\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}})$  in (2) contains all differentiable forces (both conservative and non-conservative), such as spring forces, gravitation, smooth damper forces and gyroscopic terms. Moreover, the friction forces  $\boldsymbol{\lambda}_T$  are assumed to obey Coulomb's set-valued force law:

$$\boldsymbol{\lambda}_T \in \begin{cases} -\mu |\lambda_N|, & \gamma_T > 0, \\ [-1, 1] \mu |\lambda_N|, & \gamma_T = 0, \\ \mu |\lambda_N|, & \gamma_T < 0. \end{cases} \quad (3)$$

Herein,  $\lambda_N$  is the normal force in the frictional contact,  $\mu \geq 0$  is the friction coefficient and  $\gamma_T$  is the relative sliding velocity between the bodies involved in the frictional contact ( $\gamma_T = 0$  in case of stick). The admissible values of the friction force  $\lambda_T$  form a convex set  $C_T$  which is bounded by the values of the normal force [13]:  $C_T = \{\lambda_T \mid -\mu |\lambda_N| \leq \lambda_T \leq +\mu |\lambda_N|\}$ . Now, the friction force  $\lambda_{T_i} \in C_{T_i}$  in each frictional contact  $i \in \{1, \dots, m\}$ , is described by Coulomb's set-valued force law (3). Note that the sets  $C_{T_i}$  generally depend on the normal contact force  $\lambda_{N_i} \in \mathbb{R}$ . The normal and frictional contact forces of all  $m$  contacts are gathered in columns  $\boldsymbol{\lambda}_N = \{\lambda_{N_i}\}$  and  $\boldsymbol{\lambda}_T = \{\lambda_{T_i}\}$ , respectively, and the relative sliding velocities are gathered in the column  $\boldsymbol{\gamma}_T = \{\gamma_{T_i}\}$ , for  $i = 1, \dots, m$ . We assume that these relative sliding velocities are related to the generalised velocities through:

$$\boldsymbol{\gamma}_T(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{W}_T^T(\mathbf{q}) \dot{\mathbf{q}}. \quad (4)$$

It should be noted that  $\mathbf{W}_T^T(\mathbf{q}) = \frac{\partial \boldsymbol{\gamma}_T}{\partial \dot{\mathbf{q}}}$ . This assumption is very important as it excludes rheonomic contacts.

The equation (1) or (2) together with the set-valued force law (3) for every frictional contact  $i \in \{1, \dots, m\}$  represent a differential inclusion on force-acceleration level.

This differential inclusion exhibits an equilibrium set given by

$$\mathcal{E} = \left\{ (\mathbf{q}, \dot{\mathbf{q}}) \mid \dot{\mathbf{q}} = \mathbf{0} \wedge \mathbf{h}(\mathbf{q}, \mathbf{0}) + \sum_{i=1}^m \mathbf{W}_{T_i}(\mathbf{q}) C_{T_i} \ni \mathbf{0} \right\}, \quad (5)$$

with  $\mathbf{W}_{T_i}(\mathbf{q})$  the  $i$ -th column of  $\mathbf{W}_T(\mathbf{q})$ . It should be noted that due to the fact that nonlinear mechanical systems, without dry friction, can exhibit multiple equilibria, the system with dry friction may exhibit multiple (disconnected) equilibrium sets.

Let us now state some consequences of the assumptions made, which will be used in the next section. Due to the fact that the kinetic energy can be described by  $T = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}} = \frac{1}{2} \sum_r \sum_s M_{rs} \dot{q}^r \dot{q}^s$ , with  $\mathbf{M}(\mathbf{q}) = \mathbf{M}^T(\mathbf{q})$ , we can write in tensorial language

$$\begin{aligned} T, \dot{q}^k &= \sum_r M_{kr} \dot{q}^r, \quad T, q^k = \frac{1}{2} \sum_r \sum_s \left( \frac{\partial M_{rs}}{\partial q^k} \right) \dot{q}^r \dot{q}^s, \\ \frac{d}{dt} (T, \dot{q}^k) &= \sum_r M_{kr} \ddot{q}^r + \sum_r \sum_s \left( \frac{\partial M_{kr}}{\partial q^s} \right) \dot{q}^r \dot{q}^s \\ &= \sum_r M_{kr} \ddot{q}^r + 2T, q^k \\ &\quad + \sum_r \sum_s \left( \frac{\partial M_{kr}}{\partial q^s} - \frac{\partial M_{rs}}{\partial q^k} \right) \dot{q}^r \dot{q}^s \\ \Rightarrow \frac{d}{dt} (T, \dot{\mathbf{q}}) &= \ddot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) + 2T, \mathbf{q} - (\mathbf{f}^{\text{gyr}})^T, \end{aligned} \quad (6)$$

with the gyroscopic forces [14]  $\mathbf{f}^{\text{gyr}} = \{f_k^{\text{gyr}}\}$ ,  $f_k^{\text{gyr}} = -\sum_r \sum_s \left( \frac{\partial M_{kr}}{\partial q^s} - \frac{\partial M_{rs}}{\partial q^k} \right) \dot{q}^r \dot{q}^s$ . Comparison with (1) and (2) yields

$$\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{f}^{\text{nc}}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{f}^{\text{gyr}}(\mathbf{q}, \dot{\mathbf{q}}) - (T, \mathbf{q} + U, \mathbf{q})^T. \quad (7)$$

For the stability analysis pursued in this paper it is important to note that the gyroscopic forces have zero power [14], i.e.

$$\dot{\mathbf{q}}^T \mathbf{f}^{\text{gyr}} = -\sum_k \sum_r \sum_s \left( \frac{\partial M_{kr}}{\partial q^s} - \frac{\partial M_{rs}}{\partial q^k} \right) \dot{q}^r \dot{q}^s \dot{q}^k = 0. \quad (8)$$

## III. STABILITY PROPERTIES OF EQUILIBRIUM SETS

In Section III-A, we propose sufficient conditions for the attractivity of the equilibrium set and, in Section III-B, an instability theorem for differential inclusions is proposed, which can be used to prove the instability of an equilibrium set.

### A. Attractivity of an Equilibrium Set

In this section, we will investigate the attractivity properties of the equilibrium sets defined in the previous section.

We define the following nonlinear functionals,  $\mathbb{R}^n \rightarrow \mathbb{R}$ , on  $\dot{\mathbf{q}} \in \mathbb{R}^n$ :

- $D_q^{\text{nc}}(\dot{\mathbf{q}}) := -\dot{\mathbf{q}}^T \mathbf{f}^{\text{nc}}(\mathbf{q}, \dot{\mathbf{q}})$  is the dissipation rate function of the smooth non-conservative forces  $\mathbf{f}^{\text{nc}}$ .
- $D_q^{\lambda_T}(\dot{\mathbf{q}}) := \sum_{i=1}^m -\lambda_{T_i} \gamma_{T_i}(\mathbf{q}, \dot{\mathbf{q}}) = -\boldsymbol{\gamma}_T^T \boldsymbol{\lambda}_T$  is the dissipation rate function of the frictional forces  $\boldsymbol{\lambda}_T$ .

These dissipation rate functions are functions of  $(\mathbf{q}, \dot{\mathbf{q}})$ , but we write them as nonlinear functionals on  $\dot{\mathbf{q}}$  so that we can speak of the zero-set of the functional  $D_{\dot{\mathbf{q}}}(\dot{\mathbf{q}})$ :  $D_{\dot{\mathbf{q}}}^{-1}(0) = \{\dot{\mathbf{q}} \in \mathbb{R}^n \mid D_{\dot{\mathbf{q}}}(\dot{\mathbf{q}}) = 0\}$ .

As stated before, the type of systems under investigation may exhibit multiple equilibrium sets. Here, we will study the attractivity properties of a specific given equilibrium set  $\tilde{\mathcal{E}}$ . By  $\mathbf{q}_e$  we denote an equilibrium point of the system dynamics with bilateral *frictionless* contacts ( $\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} - \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{0}$ ), from which follows that the equilibrium is determined by  $\mathbf{h}(\mathbf{q}_e, \mathbf{0}) = \mathbf{0}$ . We assume the potential energy  $U(\mathbf{q})$  to be a locally positive definite function with respect to  $\mathbf{q}_e$  with a non-vanishing gradient in a subset  $\mathcal{U}$  of the generalised coordinate space, i.e.

$$U(\mathbf{q}) = \begin{cases} 0 & \mathbf{q} = \mathbf{q}_e \\ > 0 & \forall \mathbf{q} \in \mathcal{U} \setminus \{\mathbf{q}_e\} \end{cases} \text{ and } U_{,\mathbf{q}} \neq 0, \forall \mathbf{q} \in \mathcal{U} \setminus \{\mathbf{q}_e\}. \quad (9)$$

The subset  $\mathcal{Q} = \{(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \mathbf{q} \in \mathcal{U}\}$  is assumed to enclose the equilibrium set  $\tilde{\mathcal{E}}$ . Notice that the equilibrium point  $\mathbf{q}_e$  of the system without friction is also an equilibrium point of the system with friction:  $(\mathbf{q}_e, \mathbf{0}) \in \tilde{\mathcal{E}}$ . In case the system does exhibit multiple equilibrium sets, the attractivity of  $\tilde{\mathcal{E}}$  will be only local for obvious reasons. In the following we will use the Lyapunov candidate function

$$V(\mathbf{q}, \dot{\mathbf{q}}) = T(\mathbf{q}, \dot{\mathbf{q}}) + U(\mathbf{q}), \quad (10)$$

being the sum of kinetic and potential energy. We now formulate a technical result which states conditions under which the equilibrium set can be shown to be (locally) attractive and provides an estimate for its region of attraction.

### Theorem 1 (Attractivity of an Equilibrium Set)

Consider system (2), or alternatively system (1) with friction law (3). If

- 1)  $T = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ , with  $\mathbf{M}(\mathbf{q}) = \mathbf{M}^T(\mathbf{q}) > 0$ ,
- 2) the potential energy satisfies (9) and the equilibrium set  $\tilde{\mathcal{E}}$  is the largest stationary set contained in  $\mathcal{Q}$ ; so  $\tilde{\mathcal{E}} \subset \mathcal{Q}$ ,
- 3)  $D_{\dot{\mathbf{q}}}^{\text{nc}}(\dot{\mathbf{q}}) = -\dot{\mathbf{q}}^T \mathbf{f}^{\text{nc}} \geq 0$ , i.e. the smooth non-conservative forces are dissipative, and  $\mathbf{f}^{\text{nc}} = \mathbf{0}$  for  $\dot{\mathbf{q}} = \mathbf{0}$ ,
- 4)  $D_{\dot{\mathbf{q}}}^{\text{nc}-1}(0) \cap D_{\dot{\mathbf{q}}}^{\lambda_T-1}(0) = \{\mathbf{0}\} \forall \mathbf{q}$ ,
- 5)  $\tilde{\mathcal{E}} \subset \mathcal{I}_{c^*}$  in which the set  $\mathcal{I}_{c^*}$ , with  $\mathcal{I}_c = \{(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^n \times \mathbb{R}^n \mid V(\mathbf{q}, \dot{\mathbf{q}}) < c\}$ , is the largest level set of  $V$ , given by (10), that is contained in  $\mathcal{Q}$ , i.e.  $c^* = \max_{\{c: \mathcal{I}_c \subset \mathcal{Q}\}} c$ ,

then the equilibrium set  $\tilde{\mathcal{E}}$  is locally attractive and  $\mathcal{I}_{c^*}$  is a conservative estimate for its region of attraction.

*Proof:* Note that  $V$  is positive definite around the equilibrium point  $(\mathbf{q}, \dot{\mathbf{q}}) = (\mathbf{q}_e, \mathbf{0})$  due to conditions 1 and 2 in the theorem. The time-derivative of  $V$  satisfies

$$\begin{aligned} \dot{V} &= \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + (T_{,\mathbf{q}} + U_{,\mathbf{q}}) \dot{\mathbf{q}} \\ &\stackrel{(2)}{=} \dot{\mathbf{q}}^T (\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{W}_T(\mathbf{q})\lambda_T) + (T_{,\mathbf{q}} + U_{,\mathbf{q}}) \dot{\mathbf{q}} \\ &\stackrel{(7)}{=} \dot{\mathbf{q}}^T (\mathbf{f}^{\text{nc}} + \mathbf{f}^{\text{gyr}} + \mathbf{W}_T \lambda_T) \stackrel{(4),(8)}{=} \dot{\mathbf{q}}^T \mathbf{f}^{\text{nc}} + \gamma_T^T \lambda_T. \end{aligned} \quad (11)$$

Due to conditions 3 and 4 of the theorem and the fact that  $D_{\dot{\mathbf{q}}}^{\lambda_T}(\dot{\mathbf{q}}) \leq 0$ , given the Coulomb friction law (3), it holds that  $\dot{V} = -D_{\dot{\mathbf{q}}}^{\text{nc}}(\dot{\mathbf{q}}) - D_{\dot{\mathbf{q}}}^{\lambda_T}(\dot{\mathbf{q}}) \leq 0$ . We now consider  $\dot{V}$  as a nonlinear functional on  $\dot{\mathbf{q}}$  and write

$$\dot{V} = 0 \text{ for } \dot{\mathbf{q}} \in \dot{V}_q^{-1}(0) \text{ and } \dot{V} < 0 \text{ for } \dot{\mathbf{q}} \notin \dot{V}_q^{-1}(0), \quad (12)$$

with  $\dot{V}_q^{-1}(0) = D_q^{\text{nc}-1}(0) \cap D_q^{\lambda_T-1}(0)$ . Using condition 4 of Theorem 1, it follows that  $\dot{V}_q^{-1}(0) = \{\mathbf{0}\}$  and hence

$$\dot{V} = 0, \text{ for } \dot{\mathbf{q}} = \mathbf{0} \text{ and } \dot{V} < 0, \text{ for } \dot{\mathbf{q}} \neq \mathbf{0}. \quad (13)$$

We now apply LaSalle's invariance principle, which is valid when every limit set is a positively invariant set (see the remark below). Let us hereto consider the set  $\mathcal{I}_{c^*}$  where  $c^*$  is chosen as in condition 5 of the theorem. Note that  $\mathcal{I}_{c^*}$  is a positively invariant set due to the choice of  $V$ . Moreover, the set  $\mathcal{S}$  is defined as  $\mathcal{S} = \{(\mathbf{q}, \dot{\mathbf{q}}) \mid \dot{V} = 0\} = \{(\mathbf{q}, \dot{\mathbf{q}}) \mid \dot{\mathbf{q}} = \mathbf{0}\}$ . On  $\mathcal{S}$ , the dynamics (2) of the system is described by  $\mathbf{h}(\mathbf{q}, \mathbf{0}) + \mathbf{W}_T(\mathbf{q})\lambda_T = \mathbf{0}$  and hence by the following inclusion:

$$\mathbf{h}(\mathbf{q}, \mathbf{0}) + \sum_{i=1}^m \mathbf{W}_{T_i}(\mathbf{q})C_{T_i} \ni \mathbf{0}. \quad (14)$$

Consequently, we can conclude that the largest invariant set in  $\mathcal{S} \cap \mathcal{Q}$  is the equilibrium set  $\tilde{\mathcal{E}}$  (see (5) and condition 2 of the theorem). Therefore, it can be concluded from LaSalle's invariance principle that  $\tilde{\mathcal{E}}$  is a locally attractive set and  $\mathcal{I}_{\rho^*}$  is an estimate for its region of attraction. ■

**Remark 1:** LaSalle's invariance principle is valid when every limit set is a positively invariant set [15]. A sufficient condition for the latter is continuity of the solution with respect to initial conditions. The system under consideration is a differential inclusion of Filippov-type, for which continuity with respect to the initial condition is guaranteed. Under the above assumptions, it therefore holds that every limit set is a positively invariant set and LaSalle's invariance principle can be applied.

**Remark 2:** The attractivity result in this section can be exploited to design controllers for robotic manipulators with frictional joints that guarantee attractivity of a target equilibrium set (one may have to revert to such a control objective when exact friction compensation is not possible). The control law, e.g. a simple PD controller as in [16], should, firstly, guarantee that in all generalised force directions dissipative (friction or control) forces are active (condition 3 and 4 of the theorem) and, secondly, guarantee the existence of a positive definite 'effective' potential energy function that is positive definite around the equilibrium set (conditions 2 and 5 of the theorem).

### B. Instability of Equilibrium Sets

We aim at proving the instability of equilibrium sets of mechanical systems with dry friction, under certain conditions, by showing that these equilibrium sets do not comply with the definition of Lyapunov stability, i.e. by showing that we can not find for **every**  $\epsilon$ -environment of the equilibrium set a  $\delta$ -neighbourhood of the equilibrium set such that for every



initial condition in the  $\delta$ -neighbourhood the solution will stay in the  $\epsilon$ -environment. We aim to do so by generalising the instability theorem for equilibrium points of smooth vectorfields (see e.g. [17]) to an instability theorem for equilibrium sets of differential inclusions<sup>1</sup>:

**Theorem 2 (Instability Theorem for Equilibrium Sets)**

Let  $\tilde{\mathcal{E}}$  be a bounded connected equilibrium set of the differential inclusion

$$\dot{\mathbf{x}} \in \mathbf{f}(\mathbf{x}), \text{ almost everywhere,} \quad (15)$$

with  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and where  $\mathbf{f}(\mathbf{x})$  is bounded and upper semi-continuous with a closed and (minimal) convex image. Let  $V$  be a continuously differentiable function such that  $V(\mathbf{x}_0) > V_{\mathcal{E}} > 0$  for some  $\mathbf{x}_0$ , for which  $\text{dist}(\mathbf{x}_0, \tilde{\mathcal{E}})$  is arbitrarily small, and where  $V_{\mathcal{E}} = \max_{\mathbf{x} \in \tilde{\mathcal{E}}} V(\mathbf{x})$ .

Define a set  $\mathcal{W} = \{\mathbf{x} \in \mathcal{D}_r \mid V(\mathbf{x}) \geq 0\}$ , where  $\mathcal{D}_r = \{\mathbf{x} \in \mathbb{R}^n \mid \text{dist}(\mathbf{x}, \tilde{\mathcal{E}}) \leq r\}$  is an environment of  $\tilde{\mathcal{E}}$  and choose  $r > 0$  such that  $\tilde{\mathcal{E}} \subset \mathcal{D}_r$  is the largest stationary set in  $\mathcal{D}_r$ . Now, three statements can be made:

- 1) If  $\dot{V}(\mathbf{x}) > 0$  in  $\mathcal{W} \setminus \tilde{\mathcal{E}}$ , then  $\tilde{\mathcal{E}}$  is unstable;
- 2) If  $\dot{V}(\mathbf{x}) \geq 0$  in  $\mathcal{W} \setminus \tilde{\mathcal{E}}$  and  $\tilde{\mathcal{E}} \subset \text{int}(\mathcal{W})$ , then  $\tilde{\mathcal{E}}$  is not attractive;
- 3) If  $\dot{V}(\mathbf{x}) \geq 0$  in  $\mathcal{W} \setminus \tilde{\mathcal{E}}$  and in a bounded environment of  $\tilde{\mathcal{E}}$  solutions of (15) can not stay in  $\mathcal{S} \setminus \tilde{\mathcal{E}}$  with  $\mathcal{S} = \{\mathbf{x} \in \mathbb{R}^n \mid \dot{V} = 0\}$ , then  $\tilde{\mathcal{E}}$  is unstable.

*Proof:* The point  $\mathbf{x}_0$  is in the interior of  $\mathcal{W}$  and  $V(\mathbf{x}_0) = V_{\mathcal{E}} + \delta V$  with  $\delta V > 0$ . The continuity of  $V(\mathbf{x})$  implies that  $V(\mathbf{x})$  attains its maximum  $V_{\mathcal{E}}$  in  $\tilde{\mathcal{E}}$  for a point  $\mathbf{x}^* \in \text{bdry} \tilde{\mathcal{E}}$ , because  $V(\mathbf{x}_0) > V_{\mathcal{E}}$  with  $\mathbf{x}_0$  arbitrarily close to  $\tilde{\mathcal{E}}$ .

Let us first prove statement 1 (instability) using the fact that  $\dot{V}(\mathbf{x}) > 0$  in  $\mathcal{W} \setminus \tilde{\mathcal{E}}$ : The trajectory  $\mathbf{x}(t)$  starting in  $\mathbf{x}(t_0) = \mathbf{x}_0$  must leave the set  $\mathcal{W}$ . To prove this, notice that as long as  $\mathbf{x}(t)$  is inside  $\mathcal{W}$ ,  $V(\mathbf{x}(t)) > V_{\mathcal{E}} + \delta V \forall t > t_0$  since  $\dot{V} > 0$  in  $\mathcal{W} \setminus \tilde{\mathcal{E}}$ . Note that  $\dot{V} = 0$  in  $\tilde{\mathcal{E}}$  since it is an equilibrium set. Define

$$\gamma = \min_{\mathbf{x} \in \mathcal{W}, V(\mathbf{x}) \geq V_{\mathcal{E}} + \delta V} \dot{V}(\mathbf{x}).$$

The function  $\dot{V}(\mathbf{x}) = \frac{\partial V}{\partial \mathbf{x}} \dot{\mathbf{x}} \in \frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})$  (a.e.) has a minimum on the compact set  $\{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{x} \in \mathcal{W}) \wedge (V(\mathbf{x}) \geq V_{\mathcal{E}} + \delta V)\} = \{\mathbf{x} \in \mathbb{R}^n \mid (\mathbf{x} \in \mathcal{D}_r) \wedge (V(\mathbf{x}) \geq V_{\mathcal{E}} + \delta V)\}$ . Then,  $\gamma > 0$  since  $\dot{V}(\mathbf{x}) > 0$  in  $\mathcal{W} \setminus \tilde{\mathcal{E}}$  and

$$\begin{aligned} V(\mathbf{x}(t)) &= V(\mathbf{x}_0) + \int_{t_0}^t \dot{V}(\mathbf{x}(s)) ds \\ &\geq V_{\mathcal{E}} + \delta V + \int_{t_0}^t \gamma ds \quad \forall t > t_0, \end{aligned} \quad (16)$$

$$\Rightarrow V(\mathbf{x}(t)) \geq V_{\mathcal{E}} + \delta V + \gamma(t - t_0) \quad \forall t > t_0,$$

because the set of time-instances for which  $\dot{V}(t)$  is not defined is of Lebesgue measure zero. This inequality shows

<sup>1</sup>Note that (2) and (3) together constitute a differential inclusion of the form (15).

that  $\mathbf{x}(t)$  can not stay forever in  $\mathcal{W}$  because  $V(\mathbf{x})$  is bounded on  $\mathcal{W}$ . As a consequence,  $\mathbf{x}(t)$  must leave  $\mathcal{W}$  through the surface  $\{\mathbf{x} \in \mathbb{R}^n \mid \text{dist}(\mathbf{x}, \tilde{\mathcal{E}}) = r\}$ . Namely,  $\mathbf{x}(t)$  can not leave  $\mathcal{W}$  through the surface  $V = 0$ , since  $V(\mathbf{x}(t)) > V_{\mathcal{E}} + \delta V > 0, \forall t > t_0$ . Since this can occur for  $\mathbf{x}_0$  such that  $\text{dist}(\mathbf{x}_0, \tilde{\mathcal{E}})$  is arbitrarily small, the equilibrium set  $\tilde{\mathcal{E}}$  is unstable.

Let us now prove statement 2 (exclusion of attractivity) using the fact that  $\dot{V}(\mathbf{x}) \geq 0$  in  $\mathcal{W} \setminus \tilde{\mathcal{E}}$ : repeat the reasoning above and realise that now  $\gamma \geq 0$  and thus  $V(\mathbf{x}(t)) \geq V_{\mathcal{E}} + \delta V, \forall t > t_0$ . This excludes the possibility of  $\mathbf{x}(t)$  ultimately converging to  $\tilde{\mathcal{E}}$  since, firstly,  $V \leq V_{\mathcal{E}} \forall \mathbf{x} \in \tilde{\mathcal{E}}$  and, secondly, the fact that  $\tilde{\mathcal{E}}$  is enclosed in the interior of  $\mathcal{W}$ . Since this is true for  $\mathbf{x}_0$ , for which  $\text{dist}(\mathbf{x}_0, \tilde{\mathcal{E}})$  is arbitrarily small, no neighbourhood of  $\tilde{\mathcal{E}}$  exists such that for any initial condition in this neighbourhood the solution will ultimately converge to  $\tilde{\mathcal{E}}$  as  $t \rightarrow \infty$ , i.e.  $\tilde{\mathcal{E}}$  is not attractive.

Finally, let us prove statement 3. Since solutions can not stay in  $\mathcal{S} \setminus \tilde{\mathcal{E}}, \exists t > t_0$  such that  $\mathbf{x}(t) \notin \mathcal{S}$ . Moreover, every solution  $x(t)$  of (15) is an absolutely continuous function of time and  $x(t) \notin \mathcal{S}$  for some small open time domain  $(t_0, t_1)$ . Therefore, for  $t \in (t_0, t_1), \dot{V} > 0$ . Consequently,  $\int_{t_0}^{t_1} \dot{V}(s) ds > 0$ . This implies that  $V(t)$  is strictly increasing for  $(t_0, t_1)$ . As  $t \rightarrow \infty$ , the positive contributions to  $V(t)$  will ensure that the solution will be bounded away from the equilibrium set for an initial condition arbitrarily close to the equilibrium set. As a consequence,  $\tilde{\mathcal{E}}$  is unstable. ■

#### IV. ILLUSTRATIVE EXAMPLE

We study a nonlinear mechanical system with multiple bilateral frictional constraints. Consider a rod with mass  $m$ , length  $2l$  and moment of inertia  $J_S$  around its centre of mass  $S$ , see Figure 1. The gravitational acceleration is denoted by  $g$ . The rod is subject to two holonomic bilateral constraints: Point 1 of the rod is constrained to the vertical slider and Point 2 of the rod is constrained to the horizontal slider. Coulomb friction is present in the contact between these endpoints of the rod and the grooves (friction coefficient  $\mu_1$  in the vertical slider and friction coefficient  $\mu_2$  in the horizontal slider). It should be noted that the realised friction forces depend on the constraint (normal) forces in the grooves, which in turn depend on the realised motion, i.e. on  $(\mathbf{q}, \dot{\mathbf{q}})$ . The dynamics of the system will be described in terms of the (independent) coordinate  $\theta$ , see Figure 1. The corresponding nonlinear equation of motion is given by

$$(ml^2 + J_S) \ddot{\theta} + mgl \sin \theta = 2l \sin \theta \lambda_{T_1} - 2l \cos \theta \lambda_{T_2}, \quad (17)$$

where  $\lambda_{T_1}$  and  $\lambda_{T_2}$  are the friction forces in the vertical and horizontal sliders, respectively. Equation (17) can be written in the form (2), with

$$\begin{aligned} \mathbf{M}(\mathbf{q}) &= ml^2 + J_S, \quad \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) = -mgl \sin \theta, \\ \mathbf{W}_T(\mathbf{q}) &= [2l \sin \theta \quad -2l \cos \theta]. \end{aligned} \quad (18)$$

The equilibrium set of (17) is given by (5), with  $C_{T_i} = \{\lambda_{T_i} \mid -\mu_i |\lambda_{N_i}| \leq \lambda_{T_i} \leq +\mu_i |\lambda_{N_i}|\}$ ,  $i = 1, 2$ . Note that  $C_{T_i}$ ,  $i = 1, 2$ , depend on the normal forces  $\lambda_{N_i}$ ,  $i = 1, 2$ ,

which in turn may depend on the friction forces. The static equilibrium equations of the rod yield:

$$\begin{aligned} \lambda_{N_1} + \lambda_{T_2} &= 0, \\ \lambda_{N_2} + \lambda_{T_1} - mg &= 0, \\ l \cos \theta \lambda_{N_1} - l \sin \theta \lambda_{N_2} + l \sin \theta \lambda_{T_1} - l \cos \theta \lambda_{T_2} &= 0. \end{aligned} \quad (19)$$

Based on the first two equations in (19) and Coulomb's friction law (3) for both frictional sliders, the following algebraic inclusions for the friction forces in equilibrium can be derived:

$$\begin{aligned} \lambda_{T_1} &\in [-\mu_1 |\lambda_{T_2}|, \mu_1 |\lambda_{T_2}|], \\ \lambda_{T_2} &\in [-\mu_2 |\lambda_{T_1} - mg|, \mu_2 |\lambda_{T_1} - mg|]. \end{aligned} \quad (20)$$

The resulting set of friction forces in equilibrium is depicted schematically in Figure 2. The equilibrium set  $\mathcal{E}$  in terms of the independent generalised coordinate  $\theta$  now follows from the equation of motion (17):  $mgl \sin \theta = 2l \sin \theta \lambda_{T_1} - 2l \cos \theta \lambda_{T_2}$ . For values of  $\theta$  such that  $\cos \theta \neq 0$  (we assume that, for given values for  $m$ ,  $g$  and  $l$ , the friction coefficients  $\mu_1$  and  $\mu_2$  are small enough to guarantee that this assumption is satisfied in equilibrium) we obtain:

$$\theta = \arctan \left( \frac{\lambda_{T_2}}{-\frac{mg}{2} + \lambda_{T_1}} \right) + k\pi, \quad k = 0, 1, \quad (21)$$

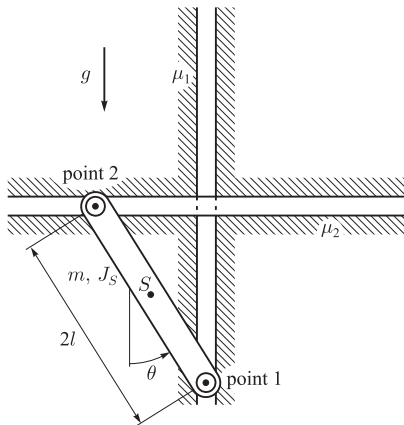


Figure 1: Rod with two frictional constraints.

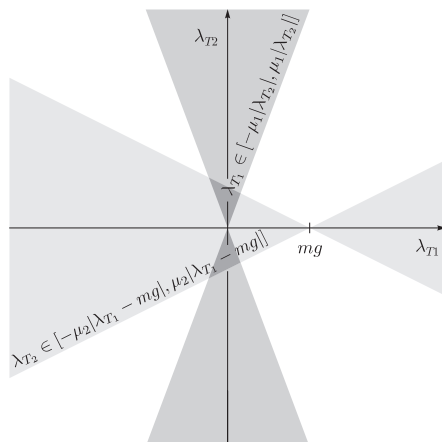


Figure 2: Attainable friction forces in equilibrium.

for values of  $\lambda_{T_1}$  and  $\lambda_{T_2}$  taken from (20). Equation (21) describes the fact that there exist two isolated equilibrium sets (an equilibrium set  $\mathcal{E}_1$  around  $\theta = 0$  and  $\mathcal{E}_2$  around  $\theta = \pi$ ) for small values of the friction coefficients. The equilibrium sets are given by

$$\mathcal{E}_k = \left\{ (\theta, \dot{\theta}) \mid \dot{\theta} = 0, -\hat{\theta} \leq \theta - (k-1)\pi \leq \hat{\theta} \right\}, \quad (22)$$

with  $\hat{\theta} = \arctan \left( \frac{2\mu_2}{1-\mu_1\mu_2} \right)$  and for  $k = 1, 2$  and  $\mu_1\mu_2 < 1$ . Note that for  $\mu_1\mu_2 \geq 1$  these isolated equilibrium sets merge into one large equilibrium set, such that any value of  $\theta$  can be attained in this equilibrium set. We will consider the case of two isolated equilibrium sets here.

Firstly, we will study the stability properties of the equilibrium set  $\mathcal{E}_1$  around  $\theta = 0$ . Let us hereto apply Theorem 1 and check the conditions stated therein. Condition 1 of this theorem is clearly satisfied since the kinetic energy is given by:  $T = \frac{1}{2} (ml^2 + J_S) \dot{\theta}^2$ . Condition 2 is also satisfied. Namely, take the set  $\mathcal{U} = \{\theta \mid |\theta| < \pi\}$  and realise that indeed the potential energy  $U = mgl(1 - \cos \theta)$  is positive definite in  $\mathcal{U}$  and  $\partial U / \partial \theta = mgl \sin \theta$  satisfies the demand  $\partial U / \partial \theta \neq 0, \forall \theta \in \mathcal{U} \setminus \{0\}$ . Since there are no smooth non-conservative forces  $D_q^{\text{nc}}(\dot{q}) = 0$ , condition 3 is satisfied. Finally, we note that  $D_q^{\text{nc}-1}(0) = \mathbb{R}$  and  $D_q^{\lambda_T-1}(0) = \{0\}$ , which implies that condition 4 of Theorem 1 is satisfied. Note, however, that the set  $\mathcal{Q} = \{(\theta, \dot{\theta}) \mid |\theta| < \pi\}$  contains the equilibrium set  $\mathcal{E}_1$  and part of the equilibrium set  $\mathcal{E}_2$  (see Figure 4). Therefore, we consider the largest level set  $V < c^*$  for which the set  $\mathcal{E}_1$  is the only equilibrium set within this level set of  $V = T + U$ . This level set is an open set and the value

$$c^* = mgl \left( 1 - \frac{1 - \mu_1\mu_2}{\sqrt{4\mu_2^2 + (1 - \mu_1\mu_2)^2}} \right) \quad (23)$$

is chosen such that its closure touches the equilibrium set  $\mathcal{E}_2$ . Consequently, we can conclude that the equilibrium set

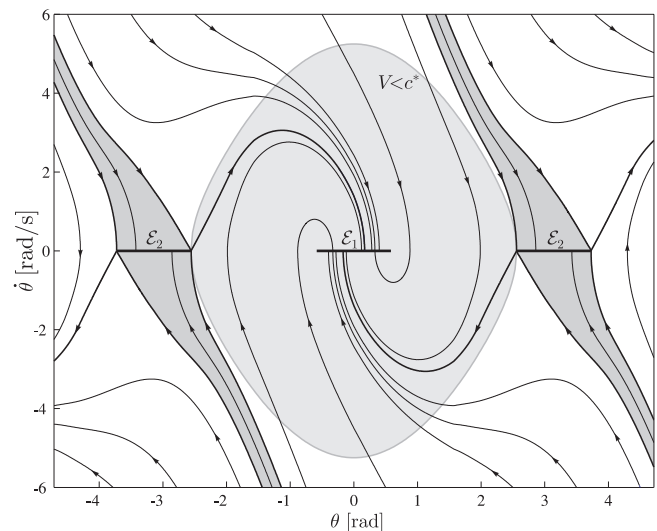


Figure 3: Phase plane and the set  $\mathcal{I}_{c^*}$  in which  $V = T + U < c^*$ .

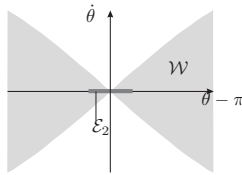


Figure 4: Schematic representation of the set  $\mathcal{W}$  in which  $V \geq 0$  ( $V$  as in (24)).

$\mathcal{E}_1$  is locally attractive. The phase plane of the constrained rod system is depicted in Figure 3 for the parameter values  $m = 1$  kg,  $J_S = \frac{1}{3}$  kg m<sup>2</sup>,  $l = 1$  m,  $\mu_1 = \mu_2 = 0.3$ ,  $g = 10$  m/s<sup>2</sup>. The trajectories in Figure 3 have been obtained numerically using the time-stepping method (see [3] and references therein). The equilibrium sets  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are indicated by thick lines on the axis  $\dot{\theta} = 0$ . It can be seen in Figure 3 that the level set  $V < c^*$  is a fairly good (though conservative) estimate for the region of attraction of the equilibrium set  $\mathcal{E}_1$ .

Secondly, we will study the stability properties of the equilibrium set  $\mathcal{E}_2$  around  $\theta = \pi$ . We apply Theorem 2 and check the conditions stated therein. The function  $V$  in this theorem is chosen as follows:

$$V = -\frac{1}{2} (J_S + ml^2) \dot{\theta}^2 + mgl(1 + \cos \theta), \quad (24)$$

and the set  $\mathcal{W} = \{(\theta, \dot{\theta}) | V \geq 0 \wedge |\theta| > \hat{\theta}\}$  is depicted schematically in Figure 4. The time-derivative of  $V$  obeys  $\dot{V} = -\dot{\theta} \mathbf{W}_T \boldsymbol{\lambda}_T = -\gamma_T^T \boldsymbol{\lambda}_T$ , with  $\mathbf{W}_T = [2l \sin \theta \quad -2l \cos \theta]$ ,  $\boldsymbol{\lambda}_T = [\lambda_{T1} \quad \lambda_{T2}]$  and  $\gamma_T = \dot{\theta} \mathbf{W}_T = [2l \dot{\theta} \sin \theta \quad -2l \dot{\theta} \cos \theta]$  are the sliding velocities in the two frictional sliders. Note that  $\dot{V} \geq 0$  for all  $(\theta, \dot{\theta}) \in \mathcal{W}$  and  $\dot{V} = 0$  if and only if  $\dot{\theta} = 0$ . Using the equation of motion (17), we can easily show that solutions can not stay in  $\mathcal{S} \setminus \mathcal{E}_2$ , with  $\mathcal{S} = \{(\theta, \dot{\theta}) | \dot{\theta} = 0\}$ , since  $\mathcal{E}_1 \cap \mathcal{W} = \emptyset$ . The conditions of statement 3 of Theorem 2 are satisfied and it can be concluded that the equilibrium set  $\mathcal{E}_2$  is unstable.

The equilibrium set  $\mathcal{E}_2$  becomes a saddle point for  $\mu_{1,2} = 0$ . This saddle structure in the phase plane (see Figure 3) remains for  $\mu_{1,2} > 0$ , but  $\mathcal{E}_2$  is a set instead of a point. Interestingly, the stable manifold of  $\mathcal{E}_2$  is ‘thick’, i.e. there exists a bundle of solutions (depicted in dark grey) which are attracted to the unstable equilibrium set  $\mathcal{E}_2$ . Put differently: the equilibrium set  $\mathcal{E}_2$  has a region of attraction, where the region is a set with a non-empty interior. The unstable half-manifolds of  $\mathcal{E}_2$  originate at the tips of the set  $\mathcal{E}_2$  and are heteroclinic orbits to the stable equilibrium set  $\mathcal{E}_1$ .

## V. CONCLUSIONS

In this paper, a class of hybrid engineering systems, namely nonlinear mechanical systems with set-valued friction, are considered. Conditions are given under which an equilibrium set of a multi-degree-of-freedom nonlinear mechanical system, with set-valued friction and an arbitrary number of frictional bilateral constraints, is attractive. The attractivity result is obtained by using the framework of differential inclusions together with a Lyapunov-type stability

analysis and LaSalle’s invariance principle. Moreover, we provide a conservative estimate for the region of attraction of the equilibrium set. Moreover, a result on the instability of equilibrium sets of differential inclusions is proposed. This result allows us to investigate the instability of equilibrium sets of nonlinear mechanical systems with frictional bilateral constraints. An example concerning a nonlinear system with two frictional bilateral constraints is studied in Section IV and the attractivity and instability of its equilibrium sets are assessed using these results.

The theorems presented in this paper have been proved for dissipative systems and form the stepping stone to the analysis of non-dissipative systems for which the equilibrium set might still be attractive due to the dissipative nature of the frictional forces (see also [12]). Future work will also involve systems with impacts (i.e. state jumps) and using the obtained results in the scope of control design for systems with set-valued friction and impact.

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