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FINDING PERIODIC SOLUTIONS OF FORCED SYSTEMS WITH LOCAL NONLINEARITIES: A MIXED SHOOTING HARMONIC BALANCE METHOD

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ABSTRACT

Several numerical approaches have been developed to capture nonlinear effects of dynamical systems. In this paper we present a mixed shooting-harmonic balance method to solve large mechanical systems with local nonlinearities efficiently. The Harmonic Balance Method as well as the shooting method have both their pros and cons. The proposed mixed shooting-HBM approach combines the efficiency of HBM and the accuracy of the shooting method and has therefore advantages of both.

INTRODUCTION

Finding periodic solutions of mechanical systems is a very important task in the design process of machines and mechanical devices. For instance, knowledge on the response of the system on harmonic excitation is essential to obtain information about high cycle fatigue behaviour. In numerous systems local nonlinearities are present due to contact or coupling elements. These local nonlinearities can have a strong impact on the global system behaviour. Therefore, the nonlinearities have to be considered in the design process and must be modeled accurately as well as in a computationally efficient way.

The most popular methods to find steady-state responses of nonlinear differential equations are the Harmonic Balance Method (HBM) [4] [5] and the Shooting Method [6]. The standard HBM approximates the periodic solution in frequency domain and is very popular as it is well suited for large systems with many states. Local nonlinearities cannot be evaluated directly in the frequency domain. The standard HBM performs an inverse Fourier transformation, and then calculates the nonlinear force in time domain and subsequently the Fourier coefficients of the nonlinear force. This procedure is often denoted as the Alternating Frequency Time Method (AFT) [3]. The disadvantage of the HBM is that strong nonlinearities are poorly represented by a truncated Fourier series. In contrast, the shooting method operates in time-domain and relies on numerical time-simulation. Set-valued force laws such as dry friction or other strong nonlinearities can be dealt with if an appropriate numerical integrator is available. The shooting method, however, becomes infeasible if the system has many states. The proposed mixed shooting-HBM approach combines the efficiency of HBM and the accuracy of the shooting method and has therefore many advantages.

In this paper the mixed shooting-HBM approach is introduced as a novel method to calculate periodic solutions of forced mechanical systems. Two different variants of the mixed shooting-HBM approach, which are called Method 1 and Method 2 in the following, are presented. Depending on the position of the local nonlinearities within the mechanical system, the one or the other is better suitable. The more general Method 2 is tested on a multi-mass oscillator with dry friction at the end of the paper and is compared to the full HBM and full shooting method.

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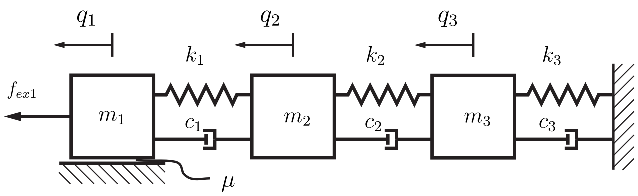

FIGURE 1: Three DOF oscillator with dry friction.

Mixed shooting-HBM approach

The mixed shooting-HBM approach uses the local character of the nonlinearities to find periodic solutions of mechanical systems efficiently. Therefore the system must be divided into linear and nonlinear subsystems. This can be done in two different ways which are defined in this paper as Method 1 and Method 2. First the system description is given and subsequently both methods are discussed.

System description

We consider a Lagrangian system

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t)+\boldsymbol{C}\dot{\boldsymbol{q}}(t)+\boldsymbol{K}\boldsymbol{q}(t)=\boldsymbol{f}_{\text{ex}}(t)+\boldsymbol{f}_{\text{nl}}(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t)) \tag{1}$$

where $\boldsymbol{f}_{\text{nl}}$ contains the nonlinear forces and $\boldsymbol{f}_{\text{ex}}(t)=\boldsymbol{f}_{\text{ex}}(t+T)$ is the periodic forcing. We assume that the system consists of three subsystems with

$$\boldsymbol{q}=\begin{pmatrix}\boldsymbol{q}_1\\ \boldsymbol{q}_2\\ \boldsymbol{q}_3\end{pmatrix}, \tag{2}$$

that the nonlinear forces only act on subsystem 1, and that the system matrices $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ have the following structure

$$\boldsymbol{M}=\begin{pmatrix}\boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \boldsymbol{M}_{13}\\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \boldsymbol{M}_{23}\\ \boldsymbol{M}_{31} & \boldsymbol{M}_{32} & \boldsymbol{M}_{33}\end{pmatrix}, \qquad \boldsymbol{f}_{\text{nl}}(\boldsymbol{q},\dot{\boldsymbol{q}})=\begin{pmatrix}\boldsymbol{f}_{\text{nl}1}(\boldsymbol{q}_1,\dot{\boldsymbol{q}}_1)\\ \boldsymbol{0}\\ \boldsymbol{0}\end{pmatrix}. \tag{3}$$

Subsystem 1 is subjected to nonlinear forces, which only depends on its own positions and velocities, and is connected to subsystem 3 through subsystem 2, e.g. the three DOF oscillator shown in Figure 1.

Method 1

This first approach can only be applied for systems which satisfy the condition

$$\boldsymbol{M}_{31}=\boldsymbol{M}_{13}=\boldsymbol{K}_{31}=\boldsymbol{K}_{13}=\boldsymbol{C}_{31}=\boldsymbol{C}_{13}=\boldsymbol{0} \tag{4}$$

and is suitable for the following relation of the dimensions of the subsystems:

$$\begin{aligned}&\dim(\boldsymbol{q}_1)>\dim(\boldsymbol{q}_2)\\ &\dim(\boldsymbol{q}_3)\gg\dim(\boldsymbol{q}_1)\\ &\dim(\boldsymbol{q}_3)\gg\dim(\boldsymbol{q}_2)\end{aligned}$$

For subsystem 2 and 3 we use a harmonic balance approach and impose (as a numerical approximation) perfect constraints on the system which force the response to be harmonic of the form

$$\boldsymbol{q}_2(t)=\hat{\boldsymbol{q}}_2^0+\sum_{k=1}^{n_H}\hat{\boldsymbol{q}}_2^{c,k}\cos k\omega t+\hat{\boldsymbol{q}}_2^{s,k}\sin k\omega t=\boldsymbol{V}_+(t)^{\mathrm{T}}\hat{\boldsymbol{q}}_2, \tag{5}$$

$$\boldsymbol{q}_3(t)=\boldsymbol{V}_+(t)^{\mathrm{T}}\hat{\boldsymbol{q}}_3, \tag{6}$$

with

$$\boldsymbol{V}_+(t)=\big(\boldsymbol{I}\ \cos(\omega t)\boldsymbol{I}\ \sin(\omega t)\boldsymbol{I}\ \ldots\ \cos(n_H\omega t)\boldsymbol{I}\ \sin(n_H\omega t)\boldsymbol{I}\big). \tag{7}$$

The Fourier coefficients are obtained from

$$\hat{\boldsymbol{q}}_i=\frac{2}{T}\int_0^T\boldsymbol{V}_-(t)\boldsymbol{q}_i(t)dt, \qquad \boldsymbol{V}_-(t)=\begin{pmatrix}\frac{1}{2}\boldsymbol{I}\\ \cos(\omega t)\boldsymbol{I}\\ \sin(\omega t)\boldsymbol{I}\\ \vdots\\ \cos(n_H\omega t)\boldsymbol{I}\\ \sin(n_H\omega t)\boldsymbol{I}\end{pmatrix}, \tag{8}$$

with $\omega=\frac{2\pi}{T}$ and n_H denotes the number of considered harmonics. The identity matrix $\boldsymbol{I}$ has here the dimension $\dim(\boldsymbol{q}_i)$. The motion $\boldsymbol{q}_1(t)$ of subsystem 1 is described in time domain and is *not* constrained to be harmonic. The equations of motion of subsystem 2 and 3 can therefore be expressed in frequency domain as

$$\begin{aligned}\boldsymbol{H}_{21}\hat{\boldsymbol{q}}_1+\boldsymbol{H}_{22}\hat{\boldsymbol{q}}_2+\boldsymbol{H}_{23}\hat{\boldsymbol{q}}_3&=\hat{\boldsymbol{f}}_{\text{ex}2},\\ \boldsymbol{H}_{32}\hat{\boldsymbol{q}}_2+\boldsymbol{H}_{33}\hat{\boldsymbol{q}}_3&=\hat{\boldsymbol{f}}_{\text{ex}3},\end{aligned} \tag{9}$$

where $\hat{\boldsymbol{q}}_1$ are the Fourier coefficients of $\boldsymbol{q}_1(t)$ and $\boldsymbol{H}_{ij}$ are the dynamic stiffness matrices

$$\boldsymbol{H}_{ij}=diag(\boldsymbol{J}_{ij,0},\boldsymbol{J}_{ij,1},\ldots\boldsymbol{J}_{ij,n}) \tag{10}$$

with

$$\boldsymbol{J}_{ij,k}=\begin{pmatrix}-\boldsymbol{M}_{ij}(k\omega)^2+\boldsymbol{K}_{ij} & \boldsymbol{C}_{ij}k\omega\\ -\boldsymbol{C}_{ij}k\omega & -\boldsymbol{M}_{ij}(k\omega)^2+\boldsymbol{K}_{ij}\end{pmatrix}. \tag{11}$$

The Fourier coefficients $\hat{\boldsymbol{q}}_3$ can be expressed in $\hat{\boldsymbol{q}}_2$ as

$$\hat{\boldsymbol{q}}_3 = \boldsymbol{H}_{33}^{-1}(\hat{\boldsymbol{f}}_{\text{ex3}} - \boldsymbol{H}_{32}\hat{\boldsymbol{q}}_2) \tag{12}$$

and can therefore be eliminated from the equations of motion in frequency domain, i.e.

$$\boldsymbol{H}_{21}\hat{\boldsymbol{q}}_1 + (\boldsymbol{H}_{22} - \boldsymbol{H}_{23}\boldsymbol{H}_{33}^{-1}\boldsymbol{H}_{32})\hat{\boldsymbol{q}}_2 = \hat{\boldsymbol{f}}_{\text{ex2}} - \boldsymbol{H}_{23}\boldsymbol{H}_{33}^{-1}\hat{\boldsymbol{f}}_{\text{ex3}}. \tag{13}$$

The equations of motion of subsystem 1 are nonlinear and are simulated in time-domain. For known $\hat{\boldsymbol{q}}_2$ one can calculate its time-domain representation $\boldsymbol{q}_2(t)$ and its derivatives and solve the differential equation for $\boldsymbol{q}_1(t)$

$$\begin{aligned}\boldsymbol{M}_{11}\ddot{\boldsymbol{q}}_1(t) + \boldsymbol{C}_{11}\dot{\boldsymbol{q}}_1(t) + \boldsymbol{K}_{11}\boldsymbol{q}_1(t) = -(\boldsymbol{M}_{12}\ddot{\boldsymbol{q}}_2(t) \\ + \boldsymbol{C}_{12}\dot{\boldsymbol{q}}_2(t) + \boldsymbol{K}_{12}\boldsymbol{q}_2(t)) + \boldsymbol{f}_{\text{ex1}}(t) + \boldsymbol{f}_{\text{nl1}}(\boldsymbol{q}_1(t), \dot{\boldsymbol{q}}_1(t))\end{aligned} \tag{14}$$

using numerical integration techniques. In particular, if the nonlinear force $\boldsymbol{f}_{nl1}$ is a dry friction force or described by another set-valued force law, then dedicated time-integration schemes such as time stepping methods [1] [2] have to be used. Here it should be noted, that the system (1) turns into a differential inclusion if a set-valued force law is considered.

A periodic solution of the system can be represented by the trajectory $\boldsymbol{q}_1(t)$ on the interval $0 \leq t \leq T$ and by the Fourier coefficients $\hat{\boldsymbol{q}}_2$, as $\hat{\boldsymbol{q}}_3$ is expressed by (12). The initial condition $\boldsymbol{q}_1(0)$ and $\dot{\boldsymbol{q}}_1(0)$ together with $\boldsymbol{q}_2(t) = \boldsymbol{V}_+(t)^{\mathrm{T}}\hat{\boldsymbol{q}}_2$ allow to construct $\boldsymbol{q}_1(t)$ over one period. The vector of unknowns

$$\boldsymbol{x} = \begin{pmatrix} \hat{\boldsymbol{q}}_2 \\ \boldsymbol{q}_1(0) \\ \dot{\boldsymbol{q}}_1(0) \end{pmatrix} \tag{15}$$

therefore fully represents a periodic solution of the system. Similar to a shooting method, we require for subsystem 1 the periodicity conditions $\boldsymbol{q}_1(T) - \boldsymbol{q}_1(0)$ and $\dot{\boldsymbol{q}}_1(T) - \dot{\boldsymbol{q}}_1(0)$, where the state at $t = T$ is obtained through numerical integration of (14). The periodicity conditions of subsystems 2 and 3 are given in frequency domain by (13) and (12). Hence, we seek a periodic solution by finding a zero of the nonlinear function

$$\boldsymbol{f}_R(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{H}_{21}\hat{\boldsymbol{q}}_1 + (\boldsymbol{H}_{22} - \boldsymbol{H}_{23}\boldsymbol{H}_{33}^{-1}\boldsymbol{H}_{32})\hat{\boldsymbol{q}}_2 - \hat{\boldsymbol{f}}_{\text{ex2}} + \boldsymbol{H}_{23}\boldsymbol{H}_{33}^{-1}\hat{\boldsymbol{f}}_{\text{ex3}} \\ \boldsymbol{q}_1(T) - \boldsymbol{q}_1(0) \\ \dot{\boldsymbol{q}}_1(T) - \dot{\boldsymbol{q}}_1(0) \end{pmatrix}. \tag{16}$$

The zeros of $\boldsymbol{f}_R(\boldsymbol{x})$ can be solved with a Newton-type method by iterating

$$\boldsymbol{x}^{i+1} = \boldsymbol{x}^i - \left(\frac{\partial \boldsymbol{f}_R}{\partial \boldsymbol{x}}\right)^{-1} \boldsymbol{f}_R(\boldsymbol{x}^i). \tag{17}$$

Method 2

Alternatively, we can divide the system only into two parts, a linear and a nonlinear subsystem, where

$$\boldsymbol{q}_L = \begin{pmatrix} \boldsymbol{q}_2 \\ \boldsymbol{q}_3 \end{pmatrix}, \qquad \boldsymbol{q}_N = \boldsymbol{q}_1. \tag{18}$$

Then the system matrices $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ and the nonlinear forces have the following structure

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{M}_{NN} & \boldsymbol{M}_{NL} \\ \boldsymbol{M}_{LN} & \boldsymbol{M}_{LL} \end{pmatrix}, \qquad \boldsymbol{f}_{nl}(\boldsymbol{q}_N, \dot{\boldsymbol{q}}_N) = \begin{pmatrix} \boldsymbol{f}_{nlN} \\ \boldsymbol{0} \end{pmatrix}. \tag{19}$$

This approach is more general than Method 1. Subsystem 1 and 3 don't have to be uncoupled since the system is not restricted to condition (4). The use of Method 2 can reduce the computational effort for systems where the relationship $\dim(\boldsymbol{q}_L) \gg \dim(\boldsymbol{q}_N)$ between the dimensions of the subsystems holds. Similar to Method 1 the linear subsystem is approximated with a truncated Fourier series

$$\begin{aligned}\boldsymbol{q}_L(t) &= \hat{\boldsymbol{q}}_L^0 + \textstyle\sum_{k=1}^{n} \hat{\boldsymbol{q}}_L^{c,k}\cos(k\omega t) + \hat{\boldsymbol{q}}_L^{s,k}\sin(k\omega t) \\ &= \boldsymbol{V}_+(t)^T\hat{\boldsymbol{q}}_L.\end{aligned} \tag{20}$$

Substituting this approximation into (1), the Fourier coefficients $\hat{\boldsymbol{q}}_L$ of the linear subsystem can be expressed in the Fourier coefficients $\hat{\boldsymbol{q}}_N$ of the nonlinear subsystem

$$\hat{\boldsymbol{q}}_L = \boldsymbol{H}_{LL}^{-1}(\hat{\boldsymbol{f}}_{\text{ex},L} - \boldsymbol{H}_{LN}\hat{\boldsymbol{q}}_N). \tag{21}$$

The equation of motion of the linear subsystem is therefore completely described by (21) and only the equation of motion of the nonlinear subsystem has to be described in the time domain. Using (21) together with (20), the time-evolution $\boldsymbol{q}_L(t)$ and its derivatives are given by $\hat{\boldsymbol{q}}_N$. Hence, a differential equation with a reduced dimension

$$\begin{aligned}\boldsymbol{M}_{NN}\ddot{\boldsymbol{q}}_N + \boldsymbol{C}_{NN}\dot{\boldsymbol{q}}_N + \boldsymbol{K}_{NN}\boldsymbol{q}_N = \\ \boldsymbol{M}_{NL}\ddot{\boldsymbol{q}}_L + \boldsymbol{C}_{NL}\dot{\boldsymbol{q}}_L + \boldsymbol{K}_{NL}\boldsymbol{q}_L - \boldsymbol{f}_{\text{ex},N} + \boldsymbol{f}_{\text{fric}}\end{aligned} \tag{22}$$

has to be solved for $\boldsymbol{q}_n(t)$ using numerical integration.

With (21) and (22) it is possible to represent a periodic solution of the full system in the unknowns

$$\boldsymbol{x} = \begin{pmatrix} \hat{\boldsymbol{q}}_N \\ \boldsymbol{q}_N(0) \\ \dot{\boldsymbol{q}}_N(0) \end{pmatrix}, \tag{23}$$

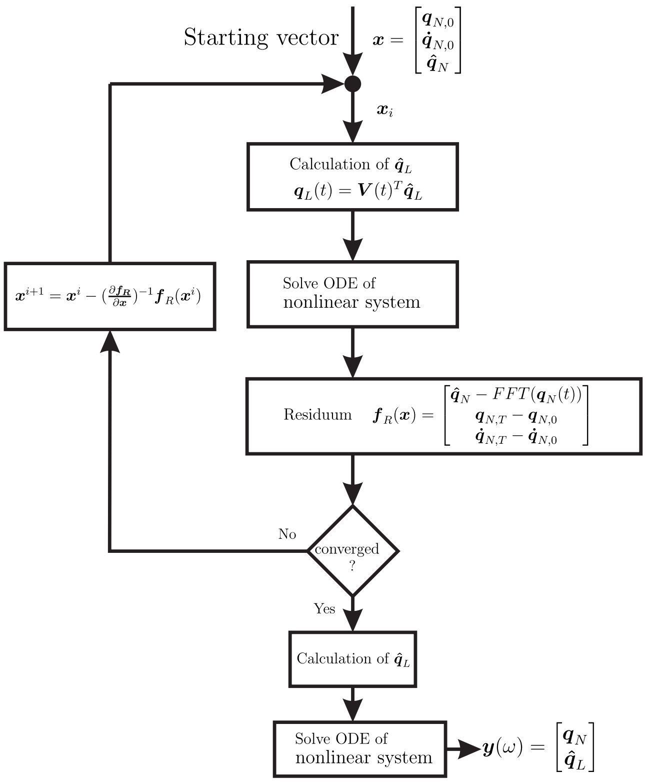

FIGURE 2: Calculation scheme of Method 2.

where $\boldsymbol{x}$ is a zero of the residuum

$$\boldsymbol{f}_R(\boldsymbol{x}) = \begin{pmatrix} \hat{\boldsymbol{q}}_N - \mathrm{FFT}(\boldsymbol{q}_N(t)) \\ \boldsymbol{q}_N(T) - \boldsymbol{q}_N(0) \\ \dot{\boldsymbol{q}}_N(T) - \dot{\boldsymbol{q}}_N(0) \end{pmatrix}. \tag{24}$$

Note that $\mathrm{FFT}(\boldsymbol{q}_N(t))$ is the Fourier transformation of the solution of the differential equation (22) and $\hat{\boldsymbol{q}}_N$ are the Fourier coefficients which represent the dynamical behaviour of the linear subsystem through (21). If $\hat{\boldsymbol{q}}_N - \mathrm{FFT}(\boldsymbol{q}_N(t)) = \mathbf{0}$ holds, then the linear subsystem is oscillating in correspondence to the movement of the nonlinear subsystem.

The iteration scheme of the mixed shooting-HBM approach (Method 2) with a Newton-type method is depicted in Figure 2. Note that, if $\dim \boldsymbol{q}_L = 0$, then the method reduces to the standard shooting approach.

Numerical comparison

The three DOF-oscillator (Figure 1) is used as a numerical benchmark to compare the mixed shooting-HBM approach (Method 2) with the full shooting method and the full HBM, in both computation effort as well as accuracy. The used parameters for the following calculations are summarized in Table 1. Since the full and the mixed shooting-HBM approach solve the nonlinear subsystem as a nonlinear differential inclusion, modern time-stepping methods with a set-valued Coulomb friction law are used for both methods. In contrast to the full and mixed shooting-HBM, the standard HBM with alternating frequency time approach only calculates the nonlinear force in time domain which makes it impossible to use the same friction model. To compare the methods in a most suitable way, the friction force for the HBM is calculated using the arctangent function. The friction force

$$f_{nl} = \mu\lambda_N \frac{2}{\pi}\arctan(\kappa\dot{q}_N) \tag{25}$$

is described with the smoothing parameter κ. The Parameters μ and λ_N are the friction coefficient and normal load, respectively. Note, that for $\lim_{\kappa\to\infty}$ the smooth friction law tends to the set-valued Coulomb friction law.

In Figure 3 the displacements of the system calculated with all three methods for the period $T = 10s$ are shown. During this period the first mass shows a pronounced stick-slip behaviour. Though for the Harmonic Balance Method 20 harmonics and for the mixed shooting-HBM only 3 harmonics are considered, the mixed method approximates much better the results of the full shooting method. The smoothing parameter is chosen preferably high ($\kappa = 800$). The mixed and full shooting method employ a set-valued description of the friction law and can therefore describe stiction precisely. The HBM, however, not only uses a smoothed friction law but also uses harmonic shape functions to approximate the friction force which leads to a poor description of this force. Contrary the mixed shooting-HBM describes the whole nonlinear subsystem in time domain and approximates only the coupling between both subsystems with harmonic shape functions.

The mixed shooting-HBM approach becomes only advantageous than the full shooting method if the dimension of the linear subsystem is much larger than that of the nonlinear subsystem. To demonstrate this, the linear subsystem is extended with additional masses. This expanded model is used to compare the full HBM, the full shooting and the mixed approach. The excitation force is chosen as $f_{\mathrm{ex}i} = 0$ for $i = 1 \ldots n-1$ and $f_{\mathrm{ex}n} = 5\cos(\omega t)$. The approximation methods are compared for one excitation frequency in computation effort and accuracy. To start the calculation for a specific excitation frequency, a starting guess for the first iteration is needed. However, the methods iterate in different unknowns and the same starting guess can therefore not be given. To provide comparable starting guesses, solutions for an excitation frequency close to the actual frequency are used as starting vectors for the iterative loops of

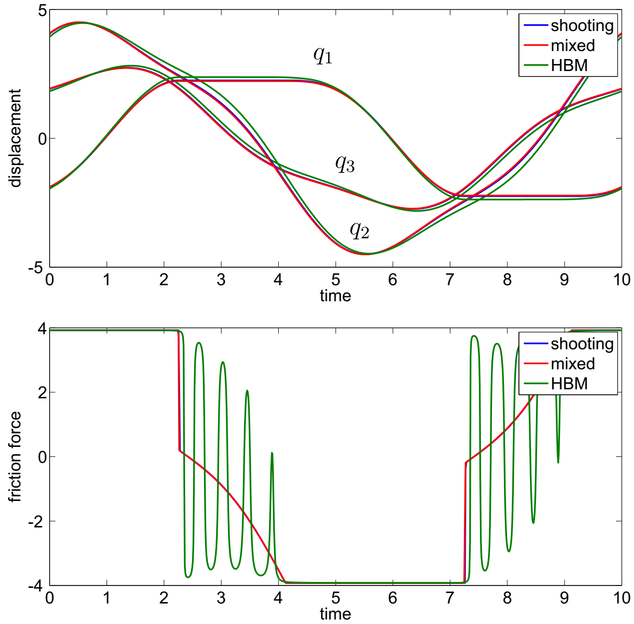

FIGURE 3: Displacement and friction force for a periodic solution with period time $T = 10s$ of the three DOF oscillator.

the respective approximation methods.

In Figure 4 the relative error of the amplitude of the first and nth mass and the calculation effort is shown for different numbers of considered harmonics n_H. Both ratios are with respect to the full shooting method, which is chosen as reference as it is almost exact.
The results show that the computation effort for a moderate accuracy can be reduced drastically by using the mixed shooting-HBM approach. Compared to the HBM, the mixed approach shows for all values of n_H more accurate results. The horizontal plateau of the relative error of the mixed method can be explained by the limited resolution of the used Fourier transformation and the integration schemes. Therefore, the increasing number of considered harmonics reduces the error only to a specific value.

Concluding Remarks

The presented mixed shooting-HBM approach shows good characteristics in accuracy as well as in calculation effort, at least for the investigated benchmark system. Depending on the system size and the nonlinear characteristics the method can be a good alternative to the commonly used methods like HBM and shooting. It should be noted, that the numerical efficiency of the methods are hard to compare and that there exist alternative HBM methods to compute periodic solutions of systems with dry friction. Further research will focus on providing a better comparison of the mixed shooting-HBM method with the existing methods.

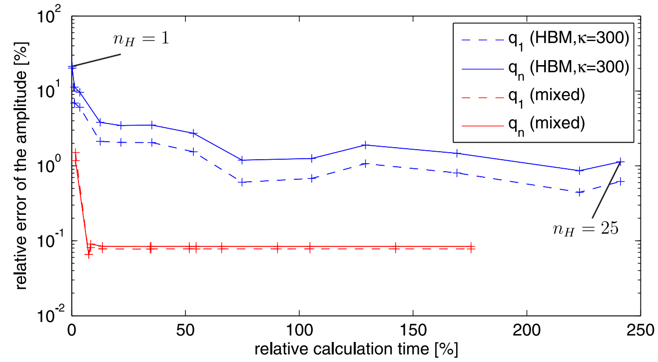

FIGURE 4: Work-precision-diagram of the HBM and the mixed shooting-HBM approach in relation to full shooting for a system of $n = 30$ masses and different numbers of considered harmonics ($n_H = 1, 3, \ldots, 25$).

TABLE 1: Selected parameters

parameter	m_i	k_i	c_i	μ	ω	$f_{ex,30}$
value	1	1	0	0.8	$\frac{1}{5}\pi$	$5\cos(\omega t)$

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