# Controlled synchronization of mechanical systems with a unilateral constraint ⋆

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**Abstract:** This paper addresses the controlled synchronization problem of mechanical systems subjected to a geometric unilateral constraint as well as the design of a switching coupling law to obtain synchronization. To define the synchronization problem, we propose a distance function induced by the quotient metric, which is based on an equivalence relation using the impact map. A Lyapunov function is constructed to investigate the synchronization problem for two identical one-dimensional mechanical systems. Sufficient conditions for the individual systems and their controlled interaction are provided under which synchronization can be ensured. We present a (coupling) control law which ensures global synchronization, also in the presence of grazing trajectories and accumulation points (Zeno behavior). The results are illustrated using a numerical example.

*Keywords:* Synchronization, measure differential inclusions, unilateral constraints, Lyapunov stability, hybrid systems

## 1. INTRODUCTION

Synchronization of coupled dynamical systems leads to 'motion in unison' which is a fundamental phenomenon appearing in, for example, biological and engineering systems. The synchronization of chaotic oscillators, neural systems and mechanical systems described by *smooth* nonlinear systems has been studied extensively, see Pikovsky et al. (2001); Nijmeijer and Rodriguez-Angeles (2003); Arenas et al. (2008); Strogatz (2003) and references therein. Synchronization of *nonsmooth* systems has received significantly less attention and to the best of the authors knowledge, the problem of synchronization for unilaterally constrained mechanical systems has not yet been addressed.

In this paper, synchronization is analyzed for mechanical systems with geometric unilateral constraints, which occur generally if mechanical systems (such as, e.g., robots) interact with a rigid environment. The dynamics of these systems comprises impacts which induce velocity jumps, rendering the system dynamics of an impulsive, hybrid nature (Leine and van de Wouw (2008); Goebel et al. (2012); Michel and Hu (1999)). For unilaterally constrained mechanical systems, accumulation points of infinitely many impact events can generally be observed, which is known as Zeno-behavior. To describe the dynamics which includes such accumulation points, system models in terms of Measure Differential Inclusions (MDIs) are employed in Moreau (1988); Leine and van de Wouw (2008).

Because impacts of unilaterally constrained mechanical systems are a consequence of collisions and therefore are state-triggered events (i.e., occur at a certain position), they generally do not occur at the same time instants for nearby trajectories. Therefore, one expects a small time-mismatch of the impact time instants even for arbitrarily close initial conditions. During this time (mismatch) interval, a large Euclidean error is observed, cf. Biemond et al. (2013); Brogliato et al. (1997); Forni et al. (2013); Leine and van de Wouw (2008); Menini and Tornambè (2001). Hence, the Euclidean synchronization error dynamics is generally unstable in the sense of Lyapunov and existing synchronization results are not applicable to mechanical systems with unilateral position constraints. An exception is the synchronization between a mechanical system and an observer, in which the impacts of the observer state can be made to coincide with the impacts of the mechanical systems, as exploited in Baumann and Leine (2015).

Recently, focusing on the stability of jumping trajectories, the 'peaking phenomenon' has been addressed for hybrid systems in the framework of Goebel et al. (2012) by considering stability in terms of a novel distance function which takes the jump characteristics into account, cf. Biemond

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et al. (2013, 2015). This approach has been extended in Postoyan et al. (2015) towards incremental stability. These approaches, however, are not applicable if either the time between state jumps can be arbitrarily small (especially in Zeno events), or if jumps can occur where the pre- and post-jump states are arbitrary close to each other. Both phenomena are generally expected in unilaterally constrained mechanical systems, motivating the synchronisation problem under study, in which hybrid trajectories are expressed by measure differential inclusions.

We distinguish three main contributions. First, we construct a distance function for mechanical systems with multiple degrees of freedom and a single geometric unilateral constraint, therewith extending the distance function design in Schatzman (1998)). This distance function can be used to define when solutions are considered close to synchronization or when they are synchronized. The synchronization problem formulation, which we establish based on the presented distance function, is applicable to generic mechanical systems with a unilateral constraint. To the best of the authors knowledge, this formulation is the first that is applicable to state-triggered hybrid systems and does not resort to Poincaré maps. Second, Lyapunov arguments are used to investigate this synchronization problem for the one-dimensional case and provide conditions on the individual systems and their controlled interaction which guarantee that synchronization indeed occurs. In contrast to the hybrid systems in Biemond et al. (2013); Forni et al. (2013), impacts with arbitrary small velocity jumps can occur, which severely complicates the Lyapunov function design and analysis. Third, we design a control law to enforce controlled synchronization using non-impulsive forces generated by the interaction network. Finally, the results are illustrated with a numerical example.

## 2. MECHANICAL SYSTEMS WITH A SINGLE UNILATERAL CONSTRAINT

We consider an $n$-DOF (degrees of freedom) mechanical system subjected to a single frictionless geometric unilateral constraint. The state of the system is described by the generalized coordinates $\boldsymbol{q}(t) \in \mathbb{R}^n$ and velocities $\boldsymbol{u}(t) \in \mathbb{R}^n$. The non-impulsive dynamics is described by the kinematic equation and the equation of motion given by

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{u}, \\ \boldsymbol{M}\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{\tau},t) &= \boldsymbol{w}\lambda, \end{aligned} \tag{1}$$

where $\boldsymbol{h}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{\tau},t)$ is a function of the state $(\boldsymbol{q},\boldsymbol{u})$, the control inputs $\boldsymbol{\tau}$ and the time $t$ explicitly. We will use the notation $(\boldsymbol{x},\boldsymbol{y}) = \left(\boldsymbol{x}^\mathsf{T}\ \boldsymbol{y}^\mathsf{T}\right)^\mathsf{T}$, where $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^n$. The mass matrix $\boldsymbol{M} = \boldsymbol{M}^\mathsf{T} \succ \boldsymbol{0}$ is symmetric and assumed to be constant and positive definite. The motion of the system is restricted by a single scleronomic geometric unilateral constraint $g(\boldsymbol{q}) \geq 0$, where $g : \mathbb{R}^n \to \mathbb{R}$ is an affine function of $\boldsymbol{q}$. The constraint velocity $\gamma(\boldsymbol{u}) = \frac{\mathrm{d}g(\boldsymbol{q}(t))}{\mathrm{d}t} = \boldsymbol{w}^\mathsf{T}\boldsymbol{u}$ is the time derivative of the constraint distance $g$, where $\boldsymbol{w} = \left(\frac{\partial g}{\partial \boldsymbol{q}}\right)^\mathsf{T}$ is the associated generalized force direction.

The force law for the constraint force $\lambda$ is described by the inequality complementarity condition, see Glocker (2001) (also referred to as Signorini's law):

$$0 \leq g(\boldsymbol{q}) \perp \lambda \geq 0, \tag{2}$$

where $a \perp b$ denotes $ab = 0$. The admissible set of states is $\mathcal{A} := \left\{(\boldsymbol{q},\boldsymbol{u}) \in \mathbb{R}^{2n} \,\middle|\, g(\boldsymbol{q}) \geq 0\right\}$. The boundary of $\mathcal{A}$ is partitioned as $\partial\mathcal{A} = \partial\mathcal{A}^+ \cup \partial\mathcal{A}^-$ with $\partial\mathcal{A}^+ := \left\{(\boldsymbol{q},\boldsymbol{u}) \in \mathbb{R}^{2n} \,\middle|\, g(\boldsymbol{q}) = 0,\, \gamma(\boldsymbol{q},\boldsymbol{u}) \geq 0\right\}$ and $\partial\mathcal{A}^- = \left\{(\boldsymbol{q},\boldsymbol{u}) \in \mathbb{R}^{2n} \,\middle|\, g(\boldsymbol{q}) = 0,\, \gamma(\boldsymbol{q},\boldsymbol{u}) < 0\right\}$. An impact is imminent if the state is in $\partial\mathcal{A}^-$ because an impact is required for the system to remain in the admissible set $\mathcal{A}$. The impulsive dynamics is described by the impact equation

$$\boldsymbol{M}(\boldsymbol{u}^+ - \boldsymbol{u}^-) = \boldsymbol{w}\Lambda, \tag{3}$$

where $\boldsymbol{u}^-(t) = \lim_{\tau\uparrow 0} \boldsymbol{u}(t+\tau)$ and $\boldsymbol{u}^+(t) = \lim_{\tau\downarrow 0} \boldsymbol{u}(t+\tau)$ are the pre- and post-impact velocities, respectively. The constraint impulse $\Lambda$ is given by the generalized Newton's law (see Glocker (2001)) with coefficient of restitution $e \in [0,1]$:

$$g(\boldsymbol{q}) = 0: \qquad 0 \leq \Lambda \perp \boldsymbol{w}^\mathsf{T}(\boldsymbol{u}^+ + e\boldsymbol{u}^-) \geq 0. \tag{4}$$

We note that infinitely many impacts can occur in a finite time interval, known as Zeno behavior or the accumulation of impact time instants. Our desire to accommodate the modeling of such behaviors motivates describing the dynamics with measure differential inclusions (1)–(4), which can be written in the compact form (see Moreau (1988); Leine and van de Wouw (2008))

$$\begin{aligned} &\mathrm{d}\boldsymbol{q} = \boldsymbol{u}\mathrm{d}t, \\ &\boldsymbol{M}\mathrm{d}\boldsymbol{u} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u},\boldsymbol{\tau},t)\mathrm{d}t = \boldsymbol{w}(\lambda\mathrm{d}t + \Lambda\mathrm{d}\eta), \end{aligned}$$

with $\lambda$ and $\Lambda$ satisfying (2) and (4). The generalized coordinates $\boldsymbol{q} : \mathbb{R} \to \mathbb{R}^n$ are absolutely continuous functions in time and their measure $\mathrm{d}\boldsymbol{q}$ has density $\boldsymbol{u}$ with respect to the Lebesgue measure $\mathrm{d}t$. The generalized velocities $\boldsymbol{u} : \mathbb{R} \to \mathbb{R}^n$ are discontinuous due to the impulsive dynamics, but they are assumed to be functions of special locally bounded variation (see Ambrosio et al. (2000)), such that the pre- and post-impact velocities $\boldsymbol{u}^-(t)$ and $\boldsymbol{u}^+(t)$, respectively, are defined for every point in time. The measure $\mathrm{d}\boldsymbol{u}$ has a density $\dot{\boldsymbol{u}}$ with respect to the Lebesgue measure $\mathrm{d}t$ and a density $(\boldsymbol{u}^+ - \boldsymbol{u}^-)$ with respect to the atomic measure $\mathrm{d}\eta$, i.e., $\mathrm{d}\boldsymbol{u} = \dot{\boldsymbol{u}}\mathrm{d}t + (\boldsymbol{u}^+ - \boldsymbol{u}^-)\mathrm{d}\eta$. The atomic measure $\mathrm{d}\eta = \sum_i \mathrm{d}\delta_{t_i}$ is the sum of Dirac point measures $\mathrm{d}\delta_{t_i}$ at the discontinuity points $t_i$, cf. Glocker (2001).

As shown in Leine and Baumann (2014), the impact equation (3) together with the impact law (4) results in an explicit impact map $\bar{Z} : (\boldsymbol{q},\boldsymbol{u}^-) \mapsto (\boldsymbol{q},\boldsymbol{u}^+) = \bar{Z}(\boldsymbol{q},\boldsymbol{u}^-)$, where

$$\begin{aligned} &\bar{Z}\left(\boldsymbol{q},\boldsymbol{u}^-\right) = \left(\boldsymbol{q}, Z_q(\boldsymbol{u}^-)\right) \\ &\text{with } Z_q(\boldsymbol{u}^-) = (1+e)\,\mathrm{prox}^{\boldsymbol{M}}_{\mathcal{T}_C(\boldsymbol{q})}(\boldsymbol{u}^-) - e\boldsymbol{u}^-, \\ &\text{where } \mathcal{T}_C(\boldsymbol{q}) = \begin{cases} \left\{\boldsymbol{u} | \boldsymbol{w}^\mathsf{T}\boldsymbol{u} \geq 0\right\} & \text{if } g(\boldsymbol{q}) = 0, \\ \mathbb{R}^n & \text{if } g(\boldsymbol{q}) > 0 \end{cases} \end{aligned} \tag{5}$$

and $\mathrm{prox}^{\boldsymbol{M}}_{\mathcal{T}}(\boldsymbol{u})$ denoting $\arg\min_{\boldsymbol{v}\in\mathcal{T}} \|\boldsymbol{u} - \boldsymbol{v}\|_{\boldsymbol{M}}$. In the following section, we consider the synchronization problem for mechanical systems of the form (1)–(4). The 'peaking phenomenon', which appears when the Euclidean synchronization error is considered, is induced by the nature of the underlying system. We construct a function $d$ that takes the role of distance and is continuous when evaluated along solutions by explicitly incorporating the impact map $\bar{Z}$. The property of non-expansivity of $\bar{Z}$ as defined in Baumann and Leine (2015) leads to a great simplification in the construction of the distance function.

## 3. SYNCHRONIZATION PROBLEM

We say that two states are synchronized if they are identical or if they are mapped to the same point in the state space by the impact map. In this sense, two points $\boldsymbol{x} = (\boldsymbol{q}_x, \boldsymbol{u}_x)$ and $\boldsymbol{y} = (\boldsymbol{q}_y, \boldsymbol{u}_y)$ in the state space are considered equivalent if they are mapped to the same point by the impact map $\bar{Z}$, which is written as

$$\boldsymbol{x} \sim \boldsymbol{y} \;\Leftrightarrow\; \bar{Z}(\boldsymbol{x}) = \bar{Z}(\boldsymbol{y}). \tag{6}$$

Similar to the synchronization manifold defined for smooth systems, we define the synchronization set as $\mathcal{S} := \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{A}^2 \,|\, \boldsymbol{x} \sim \boldsymbol{y}\}$. The synchronization set $\mathcal{S}$ can be partitioned as

$$\mathcal{S} = \mathcal{S}_{00} \cup \mathcal{S}_{01} \cup \mathcal{S}_{10} \cup \mathcal{S}_{11} \tag{7}$$

with the four subsets defined by

$$\mathcal{S}_{00} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{S} \,|\, \boldsymbol{x}, \boldsymbol{y} \in \operatorname{int}\mathcal{A} \vee \boldsymbol{x}, \boldsymbol{y} \in \partial\mathcal{A}^+\}, \tag{8}$$

$$\mathcal{S}_{01} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{S} \,|\, \boldsymbol{x} \in \partial\mathcal{A}^+ \wedge \boldsymbol{y} \in \partial\mathcal{A}^-\}, \tag{9}$$

$$\mathcal{S}_{10} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{S} \,|\, \boldsymbol{x} \in \partial\mathcal{A}^- \wedge \boldsymbol{y} \in \partial\mathcal{A}^+\}, \tag{10}$$

$$\mathcal{S}_{11} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{S} \,|\, \boldsymbol{x}, \boldsymbol{y} \in \partial\mathcal{A}^-\}. \tag{11}$$

If two states are equivalent, then either both states are in the interior $\operatorname{int}\mathcal{A}$ *or* both are on the boundary $\partial\mathcal{A}$ of $\mathcal{A}$. The partition (7)–(11) distinguishes whether two equivalent states $\boldsymbol{x}$ and $\boldsymbol{y}$ are immediately prior to an impact or not. More precisely, $\boldsymbol{x}$ has an imminent impact if $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{S}_{10} \cup \mathcal{S}_{11}$ and $\boldsymbol{y}$ has an imminent impact if $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{S}_{01} \cup \mathcal{S}_{11}$.

*Example 1.* The equivalence relation (6) and the partition (7)–(11) are illustrated using a 1-DOF mechanical system with the state vector $(q, u) \in \mathbb{R}^2$ and the single constraint $g(q) = q \geq 0$. The impact map (5) simplifies to

$$\left(q, u^+\right) = \bar{Z}\left(q, u^-\right) = \left(q, Z_q(u^-)\right)$$
$$\text{with } Z_q(u^-) = \begin{cases} -e\,u^- & \text{if } q = 0 \wedge u^- < 0, \\ u^- & \text{otherwise.} \end{cases} \tag{12}$$

A necessary condition for the equivalence of two points in the state space $\boldsymbol{x} = (q_x, u_x)$ and $\boldsymbol{y} = (q_y, u_y)$ is $q_x = q_y$, as the impact map $\bar{Z}$ does not alter the generalized coordinate. We say the unilateral constraint is called open if $g(\boldsymbol{q}) > 0$ and closed if $g(\boldsymbol{q}) = 0$. In the case of open constraints (here: $q_x = q_y > 0$), two states $\boldsymbol{x}$ and $\boldsymbol{y}$ are equivalent if and only if the velocities are identical and the synchronization set consists only of the region $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{S}_{00}$ as depicted in Fig. 1(a). The case of closed constraints (here: $q_x = q_y = 0$) is depicted in Fig. 1 for a partially elastic impact (b) and inelastic impact (c). The region $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{S}_{01}$ captures the case where $\boldsymbol{y}$ is mapped to $\boldsymbol{x}$ by the impact (i.e., $u_x = -eu_y$) and vice versa for $(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{S}_{10}$. The region $\mathcal{S}_{11}$ fills the entire quadrant $u_x < 0$, $u_y < 0$ in the case of a completely inelastic impact.

We will now introduce a notion of distance between two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in the state space in order to measure how far two solutions are away from being synchronized at a certain time $t$, similar to distance notions introduced in Biemond et al. (2015) and Burden et al. (2015); however, the distance function introduced here will exploit the properties of the impact map $\bar{Z}$. In order to avoid the 'peaking phenomenon' when evaluated along solutions, two states should also be considered close if one state has just experienced an impact and the other state is still on the verge of an impact. Using the equivalence relation (6), this can be achieved by defining the distance function $d(\boldsymbol{x}, \boldsymbol{y})$ as

$$d(\boldsymbol{x}, \boldsymbol{y}) = \inf\Big\{\textstyle\sum_{j=0}^{N} \|\boldsymbol{x}^j - \boldsymbol{y}^j\| \,\Big|\, N \in \mathbb{N}_0, \boldsymbol{x} = \boldsymbol{x}^0, \\ \boldsymbol{y}^j \sim \boldsymbol{x}^{j+1} \text{ for } 0 \leq j < N, \boldsymbol{y}^N = \boldsymbol{y}\Big\}, \tag{13}$$

where $\|\cdot\| : \boldsymbol{x} \mapsto \sqrt{\boldsymbol{x}^\mathsf{T}\boldsymbol{x}}$ denotes the Euclidean norm. The distance function $d$ is the quotient metric on the quotient space $\mathcal{A}/\sim$ obtained by the equivalence relation (6). Consequently, it satisfies the conditions of a metric on $\mathcal{A}/\!\sim$, but not on $\mathcal{A}$ itself.

*Remark 2.* The distance function $d$ serves to define the synchronization problem below and as such is needed to make explicit which system property is pursued. In contrast, the Lyapunov function which we propose in Section 4 is used to investigate this problem.

We define the synchronization problem for mechanical systems of the form (1)–(4) using the distance function $d$ defined in (13). Given two trajectories $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$, the error signal $e(t) = d(\boldsymbol{x}^+(t), \boldsymbol{y}^+(t))$ is a continuous function in time since $d(\boldsymbol{x}^-(t), \boldsymbol{y}^-(t)) = d(\boldsymbol{x}^+(t), \boldsymbol{y}^+(t))$. This observation allows us to formulate the synchronization problem as follows (cf. Blekhman et al. (1997) for definitions of synchronization for smooth differential equations).

*Definition 3.* (Synchronization problem). Consider two mechanical systems of the form (1)–(4) with solutions $\boldsymbol{x}(t) = (\boldsymbol{q}_x(t), \boldsymbol{u}_x(t))$ and $\boldsymbol{y}(t) = (\boldsymbol{q}_y(t), \boldsymbol{u}_y(t))$ for the initial conditions $\boldsymbol{x}^-(t_0)$, $\boldsymbol{y}^-(t_0) \in \mathcal{A}$. Let the inputs $\boldsymbol{\tau}_x$ and $\boldsymbol{\tau}_y$ acting on the first and second system, respectively, be defined by a static control law $(\boldsymbol{\tau}_x(t), \boldsymbol{\tau}_y(t)) = (\boldsymbol{\kappa}_x(\boldsymbol{x}(t), \boldsymbol{y}(t), t), \boldsymbol{\kappa}_y(\boldsymbol{x}(t), \boldsymbol{y}(t), t))$ and let the distance function $d$ be defined by (13). The coupled systems are said to achieve *local synchronization* if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$d(\boldsymbol{x}(t_0), \boldsymbol{y}(t_0)) < \delta(\varepsilon) \Rightarrow d(\boldsymbol{x}(t), \boldsymbol{y}(t)) < \varepsilon,\ \forall t \geq t_0 \tag{14}$$

and there exists a $\delta_0 > 0$ such that

$$d(\boldsymbol{x}(t_0), \boldsymbol{y}(t_0)) < \delta_0 \Rightarrow \lim_{t\to\infty} d(\boldsymbol{x}(t), \boldsymbol{y}(t)) = 0. \tag{15}$$

Furthermore, the coupled systems are said to achieve *global synchronization* if (14) and (15) are fulfilled and $\delta_0$ in (15) can be chosen arbitrarily large.

The distance function $d$ gives a natural notion of distance when comparing solutions and it is therefore appropriate in the definition of the synchronization problem. If two solutions $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are close at a certain point in time (i.e., $d(\boldsymbol{x}(t), \boldsymbol{y}(t))$ is small) and if the solutions are far away from the constraint, then the Euclidean distance $\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|$

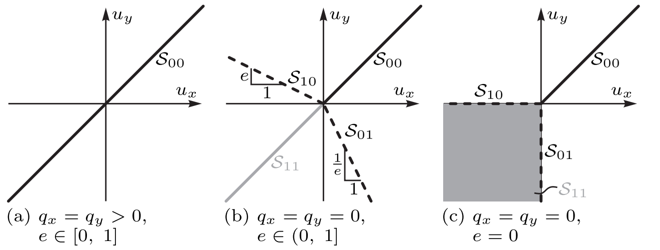


Fig. 1. Partition of the synchronization set $\mathcal{S}$ for open constraints (a), closed constraints with $e \in (0,\ 1]$ (b), and closed constraints with $e = 0$ (c).

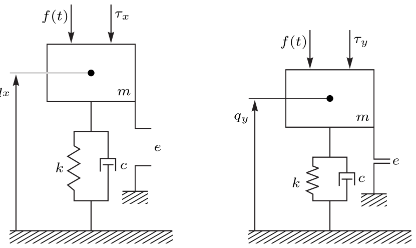


Fig. 2. Two identical unilaterally constrained 1-DOF mechanical systems subjected to an external forcing $f(t)$ and control inputs $\tau_x$ and $\tau_y$.

is small as well. The Euclidean distance might be large in the vicinity of the constraints even if the solutions are arbitrarily close to each other w.r.t. $d$. However, generally for unilaterally constrained mechanical systems, the width of the 'peaks' of the Euclidean distance tends to zero as the solutions approach each other, see Biemond et al. (2013, 2015).

To simplify the analysis of the synchronization problem, we construct a simpler (quotient) distance function $d^A(\boldsymbol{x},\boldsymbol{y})$ which is equivalent to the distance function $d$, i.e. $\alpha d^A(\boldsymbol{x},\boldsymbol{y}) \leq d(\boldsymbol{x},\boldsymbol{y}) \leq \beta d^A(\boldsymbol{x},\boldsymbol{y})$ holds for some positive scalars $\alpha$ and $\beta$. In the definition of the distance function $d$ in (13), the points $\boldsymbol{x}^{j+1} \sim \boldsymbol{y}^j$ can be seen as intermediate points and the number $N$ gives the number of these points, such that $d(\boldsymbol{x},\boldsymbol{y})$ yields the length of the shortest path from $\boldsymbol{x}$ to $\boldsymbol{y}$ via the equivalent points $\boldsymbol{x}^{j+1} \sim \boldsymbol{y}^j$, $0 \leq j < N$. At most two intermediate points are necessary in the definition of the new distance function $d^A(\boldsymbol{x},\boldsymbol{y})$ as shown in the following theorem. All proofs are omitted for the sake of brevity and can be found in Baumann et al. (2016).

*Theorem 4.* Let $d(\boldsymbol{x},\boldsymbol{y})$ be the quotient distance function in (13) with the equivalence relation (6). Then, the quotient distance function $d(\boldsymbol{x},\boldsymbol{y})$ is equivalent to $d^A(\boldsymbol{x},\boldsymbol{y})$, which is defined by

$$d^A(\boldsymbol{x},\boldsymbol{y}) := \min\left\{d^A_{00}, d^A_{01}, d^A_{10}, d^A_{11}\right\}, \tag{16}$$

where

$$d^A_{00} = \|\boldsymbol{x}-\boldsymbol{y}\|_{\boldsymbol{A}}\,, \tag{17}$$

$$d^A_{01} = \inf\left\{\left\|\boldsymbol{x}-\boldsymbol{y}^0\right\|_{\boldsymbol{A}}+\left\|\boldsymbol{x}^1-\boldsymbol{y}\right\|_{\boldsymbol{A}} \,\middle|\, \left(\boldsymbol{x}^1,\boldsymbol{y}^0\right)\in\mathcal{S}_{10}\right\}, \tag{18}$$

$$d^A_{10} = \inf\left\{\left\|\boldsymbol{x}-\boldsymbol{y}^0\right\|_{\boldsymbol{A}}+\left\|\boldsymbol{x}^1-\boldsymbol{y}\right\|_{\boldsymbol{A}} \,\middle|\, \left(\boldsymbol{x}^1,\boldsymbol{y}^0\right)\in\mathcal{S}_{01}\right\}, \tag{19}$$

$$d^A_{11} = \inf\left\{\left\|\boldsymbol{x}-\boldsymbol{y}^0\right\|_{\boldsymbol{A}}+\left\|\boldsymbol{x}^1-\boldsymbol{y}^1\right\|_{\boldsymbol{A}}+\left\|\boldsymbol{x}^2-\boldsymbol{y}\right\|_{\boldsymbol{A}} \,\middle|\, \left(\boldsymbol{x}^1,\boldsymbol{y}^0\right) \in \mathcal{S}_{01} \wedge \left(\boldsymbol{x}^2,\boldsymbol{y}^1\right) \in \mathcal{S}_{10}\right\} \tag{20}$$

with $\boldsymbol{A} = \operatorname{diag}(\boldsymbol{K},\boldsymbol{M})$, where $\boldsymbol{M}$ is the mass matrix and $\boldsymbol{K}$ is an arbitrary symmetric positive definite matrix. Furthermore, if $\boldsymbol{A}$ is the identity matrix $\boldsymbol{I}$, then $d^A(\boldsymbol{x},\boldsymbol{y}) = d^I(\boldsymbol{x},\boldsymbol{y}) = d(\boldsymbol{x},\boldsymbol{y})$.

## 4. 1-DOF MECHANICAL IMPACT OSCILLATORS

In this section, we consider 1-DOF mechanical impact oscillators as depicted in Fig. 2, which are the simplest, though relevant, representatives of the class of mechanical systems presented in Section 2. We design a synchronizing control law and construct sufficient conditions for *global* synchronization induced by the controlled interaction.

The states of the two coupled systems are denoted by $\boldsymbol{x} = (q_x, u_x)$ and $\boldsymbol{y} = (q_y, u_y)$. The equation of motion is described by (1) with $h(q,u,\tau,t) = -cu - kq - f(t) - \tau$ and $k, c > 0$. The impact equation is given by (3). Without loss of generality, we choose $m = k = 1$ as well as $w = 1$ and $g = q$. This can always be achieved using a rescaling of the states and the time. The equations of motion of the coupled system are therefore given by

$$\begin{aligned} \dot{u}_x + cu_x + q_x &= \lambda_x - f(t) - \tau_x \quad \text{with } \dot{q}_x = u_x \text{ a.e.}, \\ \dot{u}_y + cu_y + q_y &= \lambda_y - f(t) - \tau_y \quad \text{with } \dot{q}_y = u_y \text{ a.e.} \end{aligned} \tag{21}$$

The external forcing $f(t)$ is identical for both systems, whereas the control inputs $\tau_x$ and $\tau_y$ are generally unequal. Both systems are coupled if the control input $\tau_x$ depends on the state $\boldsymbol{y}$ and/or $\tau_y$ depends on $\boldsymbol{x}$. The unilateral constraints are closed if $q_x = 0$ or $q_y = 0$, respectively, and constraint forces obey Signorini's law

$$0 \leq \lambda_x \perp q_x \geq 0, \quad 0 \leq \lambda_y \perp q_y \geq 0 \tag{22}$$

and impacts are described by (3)-(4). Completely inelastic collisions are excluded, that is, the coefficient of restitution fulfills $e \in (0\ 1]$ and the explicit impact map is given as:

$$\bar{Z}\left(q_x, u_x^-\right) = \left(q_x, Z_{q_x}(u_x^-)\right), \ \bar{Z}\left(q_y, u_y^-\right) = \left(q_y, Z_{q_y}(u_y^-)\right) \tag{23}$$

with $Z_q(u^-)$ given in (12). As we are interested in the synchronization problem for the system described by (21)–(23), we aim to study the evolution of the quotient distance function $d$ defined in (13) along solutions. Since the mass is normalized to be equal to one, the matrix $\boldsymbol{A}$ in (16)–(20) can be chosen as the identity matrix. Additionally, Theorem 4 implies that the distance function $d^I$ is identical to $d$. Therefore, we can reduce the complexity of the problem by considering the simpler distance function $d^I$ instead of $d$.

In order to design the control laws

$$\tau_x(t) = \kappa_x(\boldsymbol{x}(t),\boldsymbol{y}(t),t), \quad \tau_y(t) = \kappa_y(\boldsymbol{x}(t),\boldsymbol{y}(t),t) \tag{24}$$

and to study the controlled synchronization of the system (21)–(24), we will now present a Lyapunov function suitable for investigating synchronization according to Definition 3. While this section is restricted to 1-DOF systems, the following ideas can also be used to construct a candidate Lyapunov function for mechanical systems with multiple degrees of freedom. A naive approach for a candidate Lyapunov function would be $\frac{1}{2}\left(d^I\right)^2 = \frac{1}{2}\min\left\{{d^I_{00}}^2, {d^I_{01}}^2, {d^I_{10}}^2, {d^I_{11}}^2\right\}$, necessitating differentiation of this function with respect to time. However, this approach requires explicit knowledge of the intermediate points that play a role in $d^I$ (see (18)–(20)) which have to be obtained by solving the minimization problem in the definition of $d^I_{01}$, $d^I_{10}$ and $d^I_{11}$, see (16). In order to avoid this complication and to obtain an explicit definition for a Lyapunov function, we approximate the minimizers in (18)–(20) and obtain the following candidate Lyapunov function:

$$V(\boldsymbol{x},\boldsymbol{y}) := \min\{V_{00}(\boldsymbol{x},\boldsymbol{y}), V_{01}(\boldsymbol{x},\boldsymbol{y}), V_{10}(\boldsymbol{x},\boldsymbol{y})\}, \tag{25}$$

$$\text{where } V_{00} := \tfrac{1}{2}\hat{d}_{00}^2, \ V_{01} := \tfrac{1}{2}\hat{d}_{01}^2, \ V_{10} := \tfrac{1}{2}\hat{d}_{10}^2, \tag{26}$$

$$\hat{d}_{00} := \sqrt{(q_x - q_y)^2 + (u_x - u_y)^2}, \tag{27}$$

$$\hat{d}_{01} := \begin{cases} \sqrt{(q_x + q_y)^2 + \left(\frac{q_x + q_y}{q_x + e q_y}\right)^2 (u_x + e u_y)^2} & \text{if } u_x q_y - u_y q_x > 0, \\ \sqrt{q_x^2 + u_x^2} + \sqrt{q_y^2 + u_y^2} & \text{if } u_x q_y - u_y q_x \leq 0, \end{cases} \tag{28}$$

$$\hat{d}_{10}(\boldsymbol{x},\boldsymbol{y}) := \hat{d}_{01}(\boldsymbol{y},\boldsymbol{x}). \tag{29}$$

We may write $V(\boldsymbol{x},\boldsymbol{y}) = \hat{d}(\boldsymbol{x},\boldsymbol{y})^2$ with

$$\hat{d} = \begin{cases} \min\{\hat{d}_{00}, \hat{d}_{01}\} & \text{if } u_x q_y - u_y q_x > 0, \\ \hat{d}_{00} & \text{if } u_x q_y - u_y q_x = 0, \\ \min\{\hat{d}_{00}, \hat{d}_{10}\} & \text{if } u_x q_y - u_y q_x < 0 \end{cases} \tag{30}$$

and in Baumann et al. (2016), it is shown that

$$\frac{1}{2} d^2(\boldsymbol{x},\boldsymbol{y}) \leq V(\boldsymbol{x},\boldsymbol{y}) \leq \frac{1}{2}\left(\frac{d(\boldsymbol{x},\boldsymbol{y})}{e}\right)^2. \tag{31}$$

The function $V$ in (25) is locally Lipschitz [1] in both arguments and the considered solutions are functions of special locally bounded variation. From (Leine and van de Wouw, 2008, Prop. 6.3) it follows that the candidate Lyapunov function is of special locally bounded variation as well. Therefore, the differential measure $\mathrm{d}V$ has a density $\dot{V}$ with respect to the Lebesgue measure $\mathrm{d}t$ and a density $V^+ - V^-$ with respect to the atomic measure $\mathrm{d}\eta$, i.e., $\mathrm{d}V = \dot{V}\mathrm{d}t + (V^+ - V^-)\mathrm{d}\eta$. In the following, the densities $\dot{V}$ and $(V^+ - V^-)$ are evaluated for system (21)–(23), see Lemmas 6 and 5 below, respectively, which is used later for the Lyapunov-based stability analysis.

*Lemma 5.* The Lyapunov function (25)–(29) evaluated along solutions $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$ of (21)–(23) satisfies

$$V(\boldsymbol{x}^+(t),\boldsymbol{y}^+(t)) - V(\boldsymbol{x}^-(t),\boldsymbol{y}^-(t)) \leq 0 \ \forall t.$$

The density $\dot{V}$ is generally given by $\dot{V} = \boldsymbol{\xi}^\mathsf{T}(\dot{\boldsymbol{x}},\dot{\boldsymbol{y}})$ with $\boldsymbol{\xi} \in \partial V(\boldsymbol{x},\boldsymbol{y})$, where $\partial V(\boldsymbol{x},\boldsymbol{y})$ denotes the Clarke's generalized gradient of $V(\boldsymbol{x},\boldsymbol{y})$, see Clarke (1990). In the following, we consider the three cases (i) $V_{00} < \min\{V_{01}, V_{10}\}$, (ii) $V_{01} < \min\{V_{00}, V_{10}\}$ and (iii) $V_{10} < \min\{V_{00}, V_{01}\}$. In these cases, the generalized gradient consists of a single element, that is, the gradient in the classical sense. The case for which the generalized gradient is set-valued is considered separately in the proof of Theorem 7 below.

*Lemma 6.* Let the Lyapunov function $V$ in (25)–(29) be evaluated along solutions $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$ of (21)–(23). Consider the cases (i) $V_{00} < \min\{V_{01}, V_{10}\}$, (ii) $V_{01} < \min\{V_{00}, V_{10}\}$ and (iii) $V_{10} < \min\{V_{00}, V_{01}\}$. Depending on the case, the density $\dot{V}$ is equal to

$$\text{(i) } \dot{V}_{00} = -c(u_x - u_y)^2 + (u_x - u_y)(-\tau_x + \lambda_x + \tau_y - \lambda_y), \tag{32}$$

$$\begin{aligned} \text{(ii) } \dot{V}_{01} = &-c\frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)^2 \\ &-(1-e)\frac{q_x+q_y}{q_x+eq_y}\left(1+\frac{(u_x+eu_y)^2}{(q_x+eq_y)^2}\right)(u_x q_y - q_x u_y) \\ &+\frac{(q_x+q_y)^2}{(q_x+eq_y)^2}(u_x+eu_y)((\lambda_x+e\lambda_y)-(1+e)f-(\tau_x+e\tau_y)), \end{aligned} \tag{33}$$

$$\text{(iii) } \dot{V}_{10} \text{ symmetric to case (ii).} \tag{34}$$

In the following, we will design a control law for $\tau_x$ and $\tau_y$ such that *global* synchronization is achieved also in the presence of accumulation points and grazing trajectories. The proposed control law for the control inputs $\tau_x$ and $\tau_y$ is given by (24), where

[1] The Lyapunov function is not locally Lipschitz at $q_x = q_y = 0$ with $|u_x| + |u_y| \neq 0$. However, this occurs only for a Lebesgue negligible set in time because $q_x = 0$, $|u_x| \neq 0$ as well as $q_y = 0$, $|u_y| \neq 0$ can only hold for a set in time with Lebesgue measure zero and the following conclusion still holds.

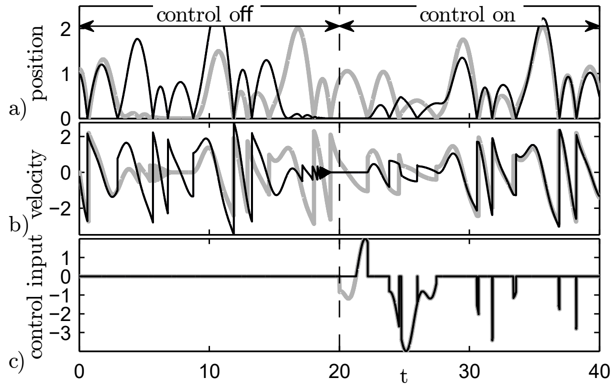


Fig. 3. a,b): Solutions $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ and control inputs $\tau_x, \tau_y$ of the 1-DOF mechanical impact oscillators for $\boldsymbol{x}$-system in black and for $\boldsymbol{y}$-system in grey, respectively. Control inputs are switched on at $t = 20$.

$$\kappa_x = \begin{cases} -f & \text{if } q_x > 0 \wedge q_y > 0 \wedge \min\{V_{01}, V_{10}\} < V_{00}, \\ -f & \text{if } q_x > 0 \wedge q_y = u_y = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{35}$$

$$\kappa_y = \begin{cases} -f & \text{if } q_x > 0 \wedge q_y > 0 \wedge \min\{V_{01}, V_{10}\} < V_{00}, \\ -f & \text{if } q_x = u_x = 0 \wedge q_y > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

Using the control law (35)–(36), the right-hand side of (21) (without impacts) becomes discontinuous. Therefore, we will consider Filippov-type solutions of system (21)–(23) together with (35)–(36).

We note that the control input vanishes if the solutions are synchronized. The proposed control law compensates the external forcing $f(t)$ whenever necessary such that the density $\dot{V}$ of the Lyapunov function (25)–(29) evaluated along solutions is non-positive. Using this control strategy, the controlled global synchronization problem is solved as shown in the following theorem.

*Theorem 7.* Let $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$ be the Filippov-type solution of system (21)–(23), where the control inputs $\tau_x$, $\tau_y$ are given by the control design (35)–(36) and let $V(\boldsymbol{x}(t),\boldsymbol{y}(t))$ be the Lyapunov function defined by (25)–(29) evaluated along the solutions. Then $\mathrm{d}V \leq 0$ and $\lim_{t\to\infty} V(\boldsymbol{x}(t),\boldsymbol{y}(t) = 0$ for all initial conditions $\boldsymbol{x}^-(t_0)$, $\boldsymbol{y}^-(t_0) \in \mathcal{A}$. Therefore, global synchronization is achieved in the sense of Definition 3.

## 5. ILLUSTRATIVE EXAMPLE

We consider system (21)–(23) with a damping constant $c = 0.01$ and a coefficient of restitution $e = 0.8$. The external forcing is chosen as $f(t) = 1+2\cos t+\cos 3t$. The controller given by the control law (35)–(36) is switched on at $t = 20$; before, the two mechanical systems are uncoupled.

The solutions $\boldsymbol{x}(t) = (q_x(t), u_x(t))$ and $\boldsymbol{y}(t) = (q_y(t), u_y(t))$ for the initial conditions $\boldsymbol{x}(t_0) = (1, -0.2)$ and $\boldsymbol{y}(t_0) = (1.1, 0.1)$ are depicted in Fig. 3. Note the accumulating impact time instants (Zeno-behavior) at $t \approx 7$ and $t \approx 18$. After the controller is switched on at $t = 20$, the distance between the solutions decreases and synchronization is achieved in accordance with Theorem 7.

Fig. 4 shows the Lyapunov function $V(\boldsymbol{x}(t),\boldsymbol{y}(t)) = \frac{1}{2}\hat{d}(\boldsymbol{x},\boldsymbol{y})$ defined by (25)–(29) (solid black line). It is con-

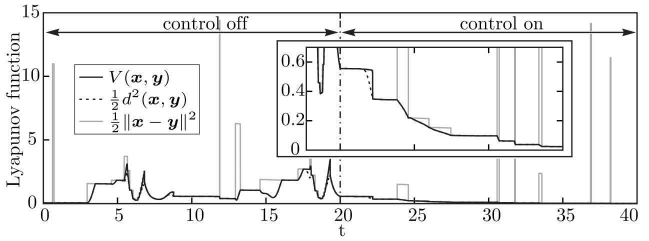


Fig. 4. The function $V(\boldsymbol{x}, \boldsymbol{y})$ (solid black) tends to zero after $t = 20$, while $V_{00}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\|\boldsymbol{x}-\boldsymbol{y}\|^2$ (gray) shows the 'peaking behavior'.

tinuous in time except when both constraints are closed at the same time, that is, when one solution has an impact and the other is in persistent contact. The (Euclidean distance) function $V_{00}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\|\boldsymbol{x}(t) - \boldsymbol{y}(t)\|^2$ (gray line) shows the undesirable 'peaking behavior' of the Euclidean synchronization error. In contrast, when the controller is switched on at $t = 20$, the Lyapunov function is a continuous monotonically decreasing function that tends to zero.

## 6. CONCLUSIONS

In this paper, we consider the controlled synchronization problem for mechanical systems with a geometric unilateral constraint inducing impacts. To define and investigate the synchronization problem for nonsmooth systems with jumping state evolutions, the Euclidean distance function is not suitable, and we resort to the quotient metric, where the equivalence relation is the equivalence kernel of the impact map. The quotient distance function is continuous in time when evaluated along solutions such that it is suitable to define stability in the sense of Lyapunov and leads to an intuitive notion of synchrony. The synchronization problem for 1-DOF forced mechanical system is investigated using Lyapunov stability analysis. The presented Lyapunov function is constructed using an approximation of the distance function.
A control law is presented which achieves global synchronization in the presence of grazing trajectories and Zeno behavior. We illustrated our results in a numerical example.

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