A mixed shooting and harmonic balance method for mechanical systems with dry friction or other local nonlinearities

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ABSTRACT

In this paper we present a mixed shooting – harmonic balance method for large linear mechanical systems with local nonlinearities. The standard harmonic balance method (HBM), which approximates the periodic solution in frequency domain, is very popular as it is well suited for large systems with many states. However, it suffers from the fact that local nonlinearities cannot be evaluated directly in the frequency domain. The standard HBM performs an inverse Fourier transform, then calculates the nonlinear force in time domain and subsequently the Fourier coefficients of the nonlinear force. The disadvantage of the HBM is, that strong nonlinearities are poorly represented by a truncated Fourier series. In contrast, the shooting method operates in time-domain and relies on numerical time-simulation. Set-valued force laws such as dry friction or other strong nonlinearities can be dealt with if an appropriate numerical integrator is available. The shooting method, however, becomes infeasible if the system has many states. The proposed mixed shooting–HBM approach combines the best of both worlds.

Keywords: Shooting method, Harmonic Balance Method, local nonlinearities, periodic solutions.

1 INTRODUCTION

Finding periodic solutions of mechanical systems is an important task in the design process of machines and mechanical devices. For instance, knowledge of the response of the system to harmonic excitation is essential to obtain information about high cycle fatigue behaviour. In engineering systems local nonlinearities are present due to contact or coupling elements. These local nonlinearities can have a strong impact on the global system behaviour. Therefore, the nonlinearities have to be considered in the design process and must be modeled accurately as well as in a computationally efficient way.

The most popular methods to find periodic steady-state responses of nonlinear differential equations are the Harmonic Balance Method (HBM) [5] [6] and the Shooting Method [7]. The standard HBM approximates the periodic solution in frequency domain and is very popular as it is well suited for large systems with many states. Local nonlinearities cannot be evaluated directly in the frequency domain. The standard HBM performs an inverse Fourier transformation, and then calculates the nonlinear force in time domain and subsequently the Fourier coefficients of the nonlinear force. This procedure is often denoted as the Alternating Frequency Time Method (AFT) [4]. The disadvantage of the HBM is that strong nonlinearities are poorly represented by a truncated Fourier series. In contrast, the shooting method operates in time-domain and relies on numerical time-simulation. Set-valued force laws such as dry friction or other strong nonlinearities can be dealt with if an appropriate numerical integrator is available. The shooting method, however, becomes infeasible if the system has many states. The proposed mixed shooting-HBM approach combines the efficiency of HBM and the accuracy of the shooting method and has therefore many advantages.

In this paper the mixed shooting-HBM approach is introduced as a novel method to calculate periodic solutions of forced mechanical systems. Two different variants of the mixed shooting-HBM approach, which are called Method 1 and Method 2 in the following, are presented. Depending on

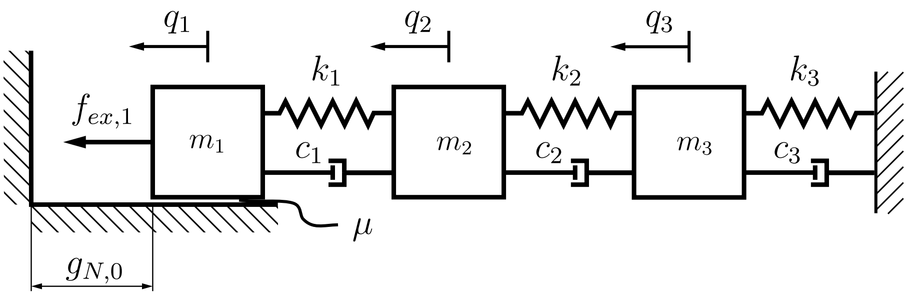


Figure 1: Three DOF oscillator with dry friction and unilateral constraint.

the position of the local nonlinearities within the mechanical system, the one or the other is better suitable. The more general Method 2 is tested on a multi-mass oscillator at the end of the paper and is compared to the full HBM and full shooting method. As local nonlinearities, dry friction as well as a hard unilateral constraint are investigated.

2 Mixed shooting-HBM approach

The mixed shooting-HBM approach uses the local character of the nonlinearities to find periodic solutions of mechanical systems efficiently. Therefore the system must be divided into linear and nonlinear subsystems. This can be done in two different ways which are defined in this paper as Method 1 and Method 2. First the system description is given and subsequently both methods are discussed.

2.1 System description

We consider a Lagrangian system of the form

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t)+\boldsymbol{C}\dot{\boldsymbol{q}}(t)+\boldsymbol{K}\boldsymbol{q}(t)=\boldsymbol{f}_{\text{ex}}(t)+\boldsymbol{f}_{\text{nl}}(\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t)), \tag{1}$$

where $\boldsymbol{f}_{\text{nl}}$ contains the nonlinear forces and $\boldsymbol{f}_{\text{ex}}(t)=\boldsymbol{f}_{\text{ex}}(t+T)$ is the periodic forcing. We assume that the system consists of three subsystems with the generalized coordinates

$$\boldsymbol{q}=\begin{pmatrix}\boldsymbol{q}_1\\ \boldsymbol{q}_2\\ \boldsymbol{q}_3\end{pmatrix}, \tag{2}$$

that the nonlinear forces only act on Subsystem 1, and that the system matrices $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ have the following structure

$$\boldsymbol{M}=\begin{pmatrix}\boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \boldsymbol{M}_{13}\\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} & \boldsymbol{M}_{23}\\ \boldsymbol{M}_{31} & \boldsymbol{M}_{32} & \boldsymbol{M}_{33}\end{pmatrix}, \qquad \boldsymbol{f}_{\text{nl}}(\boldsymbol{q},\dot{\boldsymbol{q}})=\begin{pmatrix}\boldsymbol{f}_{\text{nl1}}(\boldsymbol{q}_1,\dot{\boldsymbol{q}}_1)\\ \boldsymbol{0}\\ \boldsymbol{0}\end{pmatrix}. \tag{3}$$

Subsystem 1 is subjected to nonlinear forces, which only depend on its own positions and velocities, and is connected to Subsystem 3 through Subsystem 2, e.g. the three DOF oscillator shown in Figure 1.

2.2 Method 1

This first approach can only be applied to systems which satisfy the condition

$$\boldsymbol{M}_{31}=\boldsymbol{M}_{13}=\boldsymbol{K}_{31}=\boldsymbol{K}_{13}=\boldsymbol{C}_{31}=\boldsymbol{C}_{13}=\boldsymbol{0} \tag{4}$$

and is suitable for the following relation of the dimensions of the subsystems:

$$\dim(\boldsymbol{q}_3)\gg\dim(\boldsymbol{q}_1)>\dim(\boldsymbol{q}_2) \tag{5}$$

For Subsystem 2 and 3 we use a harmonic balance approach and impose (as a numerical approximation) perfect constraints on the system which force the response to be harmonic of the form

$$\boldsymbol{q}_2(t) = \hat{\boldsymbol{q}}_2^0 + \sum_{k=1}^{n_H} \hat{\boldsymbol{q}}_2^{c,k} \cos k\omega t + \hat{\boldsymbol{q}}_2^{s,k} \sin k\omega t = \boldsymbol{V}_+(t)^{\mathrm{T}} \hat{\boldsymbol{q}}_2, \tag{6}$$

$$\boldsymbol{q}_3(t) = \boldsymbol{V}_+(t)^{\mathrm{T}} \hat{\boldsymbol{q}}_3, \tag{7}$$

with

$$\boldsymbol{V}_+(t) = \begin{pmatrix} \boldsymbol{I} & \cos(\omega t)\boldsymbol{I} & \sin(\omega t)\boldsymbol{I} & \ldots & \cos(n_H \omega t)\boldsymbol{I} & \sin(n_H \omega t)\boldsymbol{I} \end{pmatrix}. \tag{8}$$

The Fourier coefficients of the generalized coordinates $\boldsymbol{q}_i(t)$ with $i = 1,2,3$ are obtained from

$$\hat{\boldsymbol{q}}_i = \frac{2}{T} \int_0^T \boldsymbol{V}_-(t) \boldsymbol{q}_i(t) dt, \qquad \boldsymbol{V}_-(t) = \begin{pmatrix} \frac{1}{2}\boldsymbol{I} \\ \cos(\omega t)\boldsymbol{I} \\ \sin(\omega t)\boldsymbol{I} \\ \vdots \\ \cos(n_H \omega t)\boldsymbol{I} \\ \sin(n_H \omega t)\boldsymbol{I} \end{pmatrix}, \tag{9}$$

with $\omega = \frac{2\pi}{T}$ and n_H denoting the number of considered harmonics. The identity matrix $\boldsymbol{I}$ has here the dimension $\dim(\boldsymbol{q}_i)$. The motion $\boldsymbol{q}_1(t)$ of Subsystem 1 is described in time domain and is *not* constrained to be harmonic. The equations of motion of Subsystem 2 and 3 can therefore be expressed in frequency domain as

$$\begin{aligned} \boldsymbol{H}_{21}\hat{\boldsymbol{q}}_1 + \boldsymbol{H}_{22}\hat{\boldsymbol{q}}_2 + \boldsymbol{H}_{23}\hat{\boldsymbol{q}}_3 &= \hat{\boldsymbol{f}}_{\mathrm{ex2}}, \\ \boldsymbol{H}_{32}\hat{\boldsymbol{q}}_2 + \boldsymbol{H}_{33}\hat{\boldsymbol{q}}_3 &= \hat{\boldsymbol{f}}_{\mathrm{ex3}}, \end{aligned} \tag{10}$$

where $\boldsymbol{H}_{ij}$ are the dynamic stiffness matrices

$$\boldsymbol{H}_{ij} = diag(\boldsymbol{J}_{ij,0}, \boldsymbol{J}_{ij,1}, \ldots \boldsymbol{J}_{ij,n_H}) \tag{11}$$

with

$$\boldsymbol{J}_{ij,k} = \begin{pmatrix} -\boldsymbol{M}_{ij}(k\omega)^2 + \boldsymbol{K}_{ij} & \boldsymbol{C}_{ij}k\omega \\ -\boldsymbol{C}_{ij}k\omega & -\boldsymbol{M}_{ij}(k\omega)^2 + \boldsymbol{K}_{ij} \end{pmatrix}. \tag{12}$$

Using (10) the Fourier coefficients $\hat{\boldsymbol{q}}_3$ can be expressed in $\hat{\boldsymbol{q}}_2$ as

$$\hat{\boldsymbol{q}}_3 = \boldsymbol{H}_{33}^{-1}(\hat{\boldsymbol{f}}_{\mathrm{ex3}} - \boldsymbol{H}_{32}\hat{\boldsymbol{q}}_2) \tag{13}$$

and can therefore be eliminated from the equations of motion in frequency domain, i.e.

$$\boldsymbol{H}_{21}\hat{\boldsymbol{q}}_1 + (\boldsymbol{H}_{22} - \boldsymbol{H}_{23}\boldsymbol{H}_{33}^{-1}\boldsymbol{H}_{32})\hat{\boldsymbol{q}}_2 = \hat{\boldsymbol{f}}_{\mathrm{ex2}} - \boldsymbol{H}_{23}\boldsymbol{H}_{33}^{-1}\hat{\boldsymbol{f}}_{\mathrm{ex3}}. \tag{14}$$

The equations of motion of Subsystem 1 are nonlinear and are simulated in time-domain. For known $\hat{\boldsymbol{q}}_2$ one can calculate its time-domain representation $\boldsymbol{q}_2(t)$ and its derivatives and solve the differential equation for $\boldsymbol{q}_1(t)$

$$\begin{aligned} \boldsymbol{M}_{11}\ddot{\boldsymbol{q}}_1(t) + \boldsymbol{C}_{11}\dot{\boldsymbol{q}}_1(t) + \boldsymbol{K}_{11}\boldsymbol{q}_1(t) = -(&\boldsymbol{M}_{12}\ddot{\boldsymbol{q}}_2(t) \\ &+ \boldsymbol{C}_{12}\dot{\boldsymbol{q}}_2(t) + \boldsymbol{K}_{12}\boldsymbol{q}_2(t)) + \boldsymbol{f}_{\mathrm{ex1}}(t) + \boldsymbol{f}_{\mathrm{nl1}}(\boldsymbol{q}_1(t), \dot{\boldsymbol{q}}_1(t)) \end{aligned} \tag{15}$$

using numerical integration techniques. In particular, if the nonlinear force $\boldsymbol{f}_{nl1}$ is a dry friction force or, more generally, described by a set-valued force law, then dedicated time-integration schemes such as timestepping methods [1] [3] have to be used. Here it should be noted, that the system (1) and consequently (15) turns into a differential inclusion if a set-valued force law is considered.

A periodic solution of the system can be represented by the trajectory $\boldsymbol{q}_1(t)$ on the interval $0 \leq t \leq T$ and by the Fourier coefficients $\hat{\boldsymbol{q}}_2$, as $\hat{\boldsymbol{q}}_3$ is expressed by (13). The initial condition $\boldsymbol{q}_1(0)$ and $\dot{\boldsymbol{q}}_1(0)$ together with $\boldsymbol{q}_2(t) = \boldsymbol{V}_+(t)^{\mathrm{T}}\hat{\boldsymbol{q}}_2$ allow to construct $\boldsymbol{q}_1(t)$ over one period. The vector of unknowns

$$\boldsymbol{x} = \begin{pmatrix} \hat{\boldsymbol{q}}_2 \\ \boldsymbol{q}_1(0) \\ \dot{\boldsymbol{q}}_1(0) \end{pmatrix} \tag{16}$$

therefore fully represents a periodic solution of the system. Similar to a shooting method, we require for Subsystem 1 the periodicity conditions $\boldsymbol{q}_1(T) - \boldsymbol{q}_1(0) = \boldsymbol{0}$ and $\dot{\boldsymbol{q}}_1(T) - \dot{\boldsymbol{q}}_1(0) = \boldsymbol{0}$, where the state at $t = T$ is obtained through numerical time-integration of (15). The periodicity conditions of Subsystems 2 and 3 are given in frequency domain by (14) and (13). Hence, we seek a periodic solution by finding a zero of the nonlinear function

$$\boldsymbol{f}_R(\boldsymbol{x}) = \begin{pmatrix} \boldsymbol{H}_{21}\hat{\boldsymbol{q}}_1 + (\boldsymbol{H}_{22} - \boldsymbol{H}_{23}\boldsymbol{H}_{33}^{-1}\boldsymbol{H}_{32})\hat{\boldsymbol{q}}_2 - \hat{\boldsymbol{f}}_{\mathrm{ex}2} + \boldsymbol{H}_{23}\boldsymbol{H}_{33}^{-1}\hat{\boldsymbol{f}}_{\mathrm{ex}3} \\ \boldsymbol{q}_1(T) - \boldsymbol{q}_1(0) \\ \dot{\boldsymbol{q}}_1(T) - \dot{\boldsymbol{q}}_1(0) \end{pmatrix}. \tag{17}$$

The zeros of $\boldsymbol{f}_R(\boldsymbol{x})$ can be solved with a Newton-type method by iterating

$$\boldsymbol{x}^{i+1} = \boldsymbol{x}^i - \left(\frac{\partial \boldsymbol{f}_R}{\partial \boldsymbol{x}}\right)^{-1} \boldsymbol{f}_R(\boldsymbol{x}^i). \tag{18}$$

2.3 Method 2

Alternatively, we can divide the system only into two parts, a linear and a nonlinear subsystem, where

$$\boldsymbol{q}_L = \begin{pmatrix} \boldsymbol{q}_2 \\ \boldsymbol{q}_3 \end{pmatrix}, \qquad \boldsymbol{q}_N = \boldsymbol{q}_1. \tag{19}$$

The system matrices $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ and the nonlinear forces have then the following structure

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{M}_{NN} & \boldsymbol{M}_{NL} \\ \boldsymbol{M}_{LN} & \boldsymbol{M}_{LL} \end{pmatrix}, \qquad \boldsymbol{f}_{nl}(\boldsymbol{q}_N, \dot{\boldsymbol{q}}_N) = \begin{pmatrix} \boldsymbol{f}_{nlN} \\ \boldsymbol{0} \end{pmatrix}. \tag{20}$$

This approach is more general than Method 1. Subsystem 1 and 3 do not have to be uncoupled since the system is not restricted to condition (4). The use of Method 2 can reduce the computational effort for systems for which the relationship $\dim(\boldsymbol{q}_L) \gg \dim(\boldsymbol{q}_N)$ between the dimensions of the subsystems holds. Similar to Method 1, the motion of the linear subsystem is approximated by a truncated Fourier series

$$\boldsymbol{q}_L(t) = \hat{\boldsymbol{q}}_L^0 + \sum_{k=1}^{n_H} \hat{\boldsymbol{q}}_L^{c,k}\cos(k\omega t) + \hat{\boldsymbol{q}}_L^{s,k}\sin(k\omega t) = \boldsymbol{V}_+(t)^T\hat{\boldsymbol{q}}_L. \tag{21}$$

Substituting this approximation into (1), the Fourier coefficients $\hat{\boldsymbol{q}}_L$ of the linear subsystem can be expressed in the Fourier coefficients $\hat{\boldsymbol{q}}_N$ of the nonlinear subsystem

$$\hat{\boldsymbol{q}}_L = \boldsymbol{H}_{LL}^{-1}(\hat{\boldsymbol{f}}_{\mathrm{ex},L} - \boldsymbol{H}_{LN}\hat{\boldsymbol{q}}_N). \tag{22}$$

The equation of motion of the linear subsystem is therefore completely described by (22) and only the equation of motion of the nonlinear subsystem has to be described in the time domain. Using (22) together with (21), the time-evolution $\boldsymbol{q}_L(t)$ and its derivatives are given by $\hat{\boldsymbol{q}}_N$. Hence, a differential equation with a reduced dimension

$$\boldsymbol{M}_{NN}\ddot{\boldsymbol{q}}_N + \boldsymbol{C}_{NN}\dot{\boldsymbol{q}}_N + \boldsymbol{K}_{NN}\boldsymbol{q}_N = \boldsymbol{M}_{NL}\ddot{\boldsymbol{q}}_L + \boldsymbol{C}_{NL}\dot{\boldsymbol{q}}_L + \boldsymbol{K}_{NL}\boldsymbol{q}_L - \boldsymbol{f}_{\mathrm{ex},N} + \boldsymbol{f}_{\mathrm{fric}} \tag{23}$$

has to be solved for $\boldsymbol{q}_N(t)$ using numerical time integration.

With (22) and (23) it is possible to represent a periodic solution of the full system in the unknowns

$$\boldsymbol{x} = \begin{pmatrix} \hat{\boldsymbol{q}}_N \\ \boldsymbol{q}_N(0) \\ \dot{\boldsymbol{q}}_N(0) \end{pmatrix}, \tag{24}$$

where $\boldsymbol{x}$ is a zero of the residuum

$$\boldsymbol{f}_R(\boldsymbol{x}) = \begin{pmatrix} \hat{\boldsymbol{q}}_N - \mathrm{FFT}(\boldsymbol{q}_N(t)) \\ \boldsymbol{q}_N(T) - \boldsymbol{q}_N(0) \\ \dot{\boldsymbol{q}}_N(T) - \dot{\boldsymbol{q}}_N(0) \end{pmatrix}. \tag{25}$$

Note that $\mathrm{FFT}(\boldsymbol{q}_N(t))$ is the Fourier transformation (9) of the solution of the differential equation (23) and $\hat{\boldsymbol{q}}_N$ are the Fourier coefficients which represent the dynamical behaviour of the linear subsystem through (22). If $\hat{\boldsymbol{q}}_N - \mathrm{FFT}(\boldsymbol{q}_N(t)) = \boldsymbol{0}$ holds, then the linear subsystem is oscillating in correspondence to the movement of the nonlinear subsystem.

The iteration scheme of the mixed shooting-HBM approach (Method 2) with a Newton-type method is depicted in Figure 2. Note that, if $\dim(\boldsymbol{q}_L) = 0$, then the method reduces to the standard shooting approach.

3 Numerical examples

The three DOF-oscillator (Figure 1) is used as a numerical benchmark to compare the mixed shooting-HBM approach (Method 2) with the full shooting method and the full HBM, in both computation effort as well as accuracy.
Since the full and the mixed shooting-HBM approach solve the nonlinear subsystem as a nonlinear differential inclusion, modern time-stepping methods with a set-valued force law are used for both methods. In contrast to the full and mixed shooting-HBM, the standard HBM with alternating frequency time approach only calculates the nonlinear force in time domain which makes it impossible to use the same contact model. Two types of contacts are considered separately in this work to compare the different methods for a system which is subject to friction or to a completely elastic unilateral constraint.

3.1 System with friction

First, the different methods are investigated for a system under influence of dry friction. Using the mixed shooting-HBM or the shooting approach a set-valued force law can be used within the concept of (measure) differential inclusions. The friction force is expressed by the set-valued relationship

$$-\lambda_T \in \begin{cases} \mu F_N, & \gamma_T > 0, \\ [-1,1]\,\mu F_N, & \gamma_T = 0, \\ -\mu F_N, & \gamma_T < 0. \end{cases} \tag{26}$$

The parameters μ and F_N are the friction coefficient and normal load, respectively. This friction model cannot be used for the HBM because the problem is not solved in time domain. To compare the methods in a most suitable way, the friction force for the HBM is approximated using an arctangent function

$$-\lambda_T^{smooth} = \mu F_N \frac{2}{\pi} \arctan(\kappa \gamma_T), \tag{27}$$

being a smoothed approximation of (26). The approximation (27) tends to the set-valued force law (26) for large values of the smoothing parameter κ, see Figure 3.

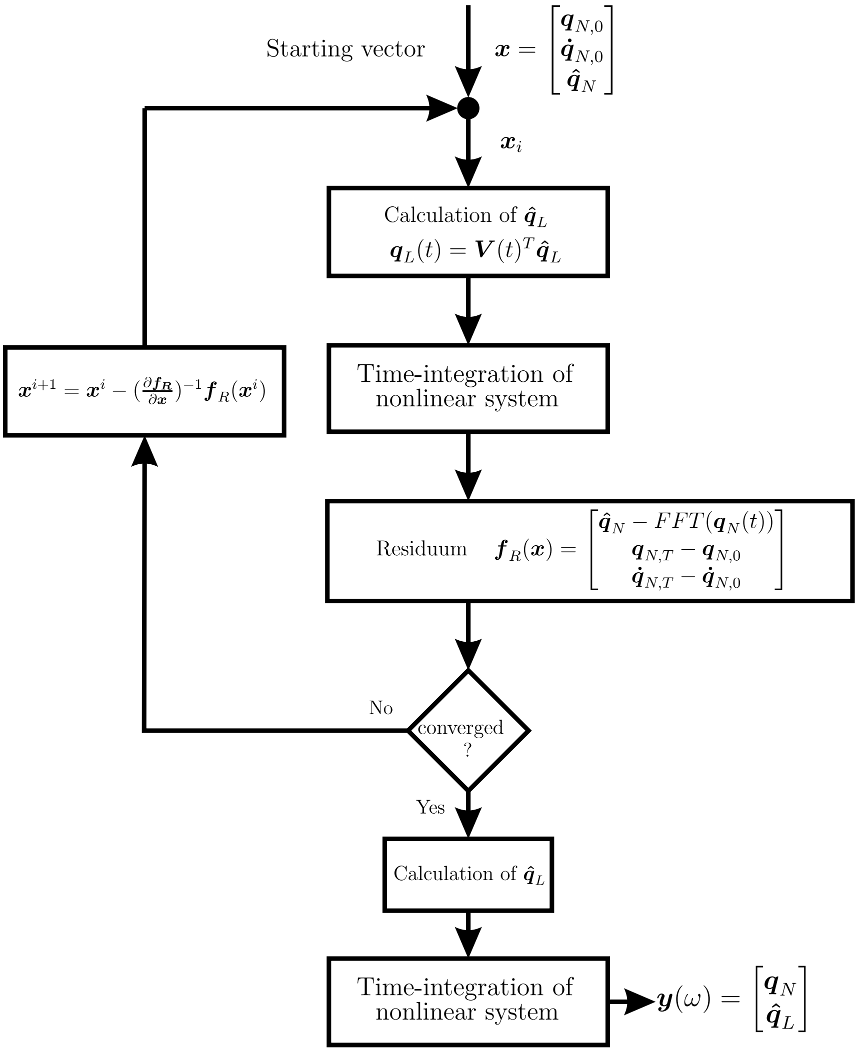


Figure 2: Calculation scheme of mixed shooting-HBM (Method 2).

In Figure 4 the displacements of the system calculated with all three methods for the period $T = 10s$ are shown. During this period the first mass shows a pronounced stick-slip behaviour. Though for the Harmonic Balance Method 20 harmonics and for the mixed shooting-HBM only 3 harmonics are considered, the mixed method approximates much better the results of the full shooting method. The smoothing parameter is chosen preferably high ($\kappa = 800$). The mixed and full shooting method employ the set-valued description (26) of the friction law and can therefore describe stiction precisely. The HBM, however, not only uses the smoothed friction law (27) but also uses harmonic shape functions to approximate the friction force which leads to a poor description of this force. In contrast, the mixed shooting-HBM describes the whole nonlinear subsystem in time domain and approximates only the coupling between both subsystems with harmonic shape functions.

The mixed shooting-HBM approach becomes more advantageous than the full shooting method if the dimension of the linear subsystem is much larger than that of the nonlinear subsystem. To demonstrate this, the linear subsystem is extended with additional masses. This expanded model is used to compare the full HBM, the full shooting and the mixed approach. The excitation force is chosen as $f_{\mathrm{ex}i} = 0$ for $i = 1 \ldots n-1$ and $f_{\mathrm{ex}n} = 5\cos(\omega t)$. The methods are compared for one excitation frequency in computation effort and accuracy. To start the calculation for a specific excitation frequency, a starting guess for the first iteration is needed. However, the methods iterate in

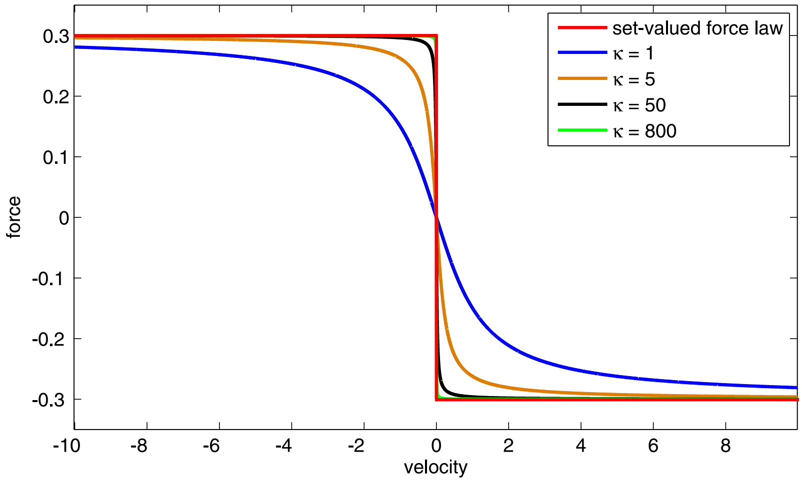


Figure 3: Set-valued friction force and approximated friction force for different values of the smoothing parameter κ.

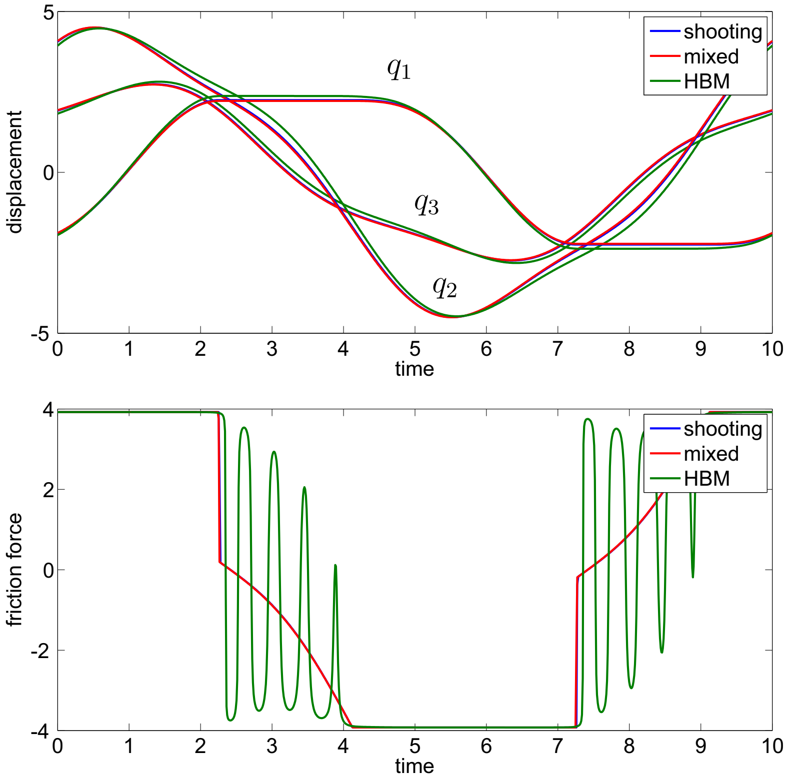


Figure 4: Displacement and friction force for a periodic solution with period time $T = 10s$ of the three DOF oscillator with dry friction.

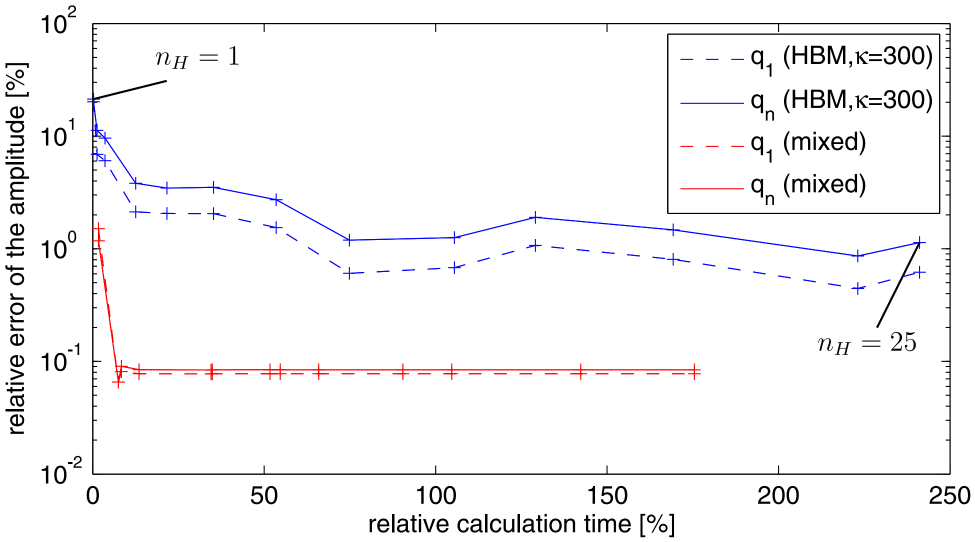


Figure 5: Work-precision-diagram of the HBM and the mixed shooting-HBM approach in relation to full shooting for a system of $n = 30$ masses with friction and different numbers of considered harmonics ($n_H = 1, 3, \ldots, 25$).

different unknowns and the same starting guess can therefore not be given. To provide comparable starting guesses, solutions for an excitation frequency close to the actual frequency are used as starting vectors for the iterative loops of the respective approximation methods.

In Figure 5 the relative error of the amplitude of the first and nth mass and the calculation effort is shown for different numbers of considered harmonics n_H. Both ratios are with respect to the full shooting method, which is chosen as reference as it is almost exact.
The results show that the computation effort for a moderate accuracy can be reduced drastically by using the mixed shooting-HBM approach. Compared to the HBM, the mixed approach shows for all values of n_H more accurate results. The horizontal plateau of the relative error of the mixed method can be explained by the limited resolution of the used Fourier transformation and the integration schemes. Therefore, the increasing number of considered harmonics reduces the error only to a specific value.

The used parameters for the calculations in this chapter are summarized in Table 1.

Table 1: Selected parameters for the system with friction.

parameter	m_i	k_i	c_i	μ	ω	$f_{ex,30}$
value	1	1	0	0.8	$\frac{1}{5}\pi$	$5\cos(\omega t)$

3.2 System with unilateral constraint

In the second example, the friction force in the first mass is replaced by a unilateral constraint. The unilateral constraint is modeled within the concept of measure differential inclusions using the hard contact law

$$0 \leq g_N \perp \lambda_N \geq 0, \tag{28}$$

where g_N is the gap ($g_N = g_{N,0} - q_1$) and λ_N represents the contact force. The Newtonian impact law is expressed through the inequality complementarity

$$0 \leq \gamma_N^+ + e_N \gamma_N^- \perp \Lambda_N \geq 0 \quad \text{with } 0 \leq e_N \leq 1, \tag{29}$$

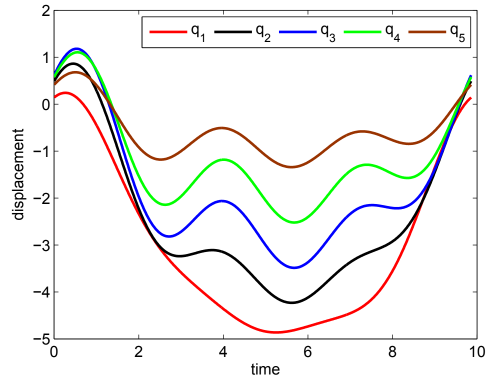
with the post-and pre-impact relative velocities γ_N^+ and γ_N^-, the contact impulse Λ_N and the restitution coefficient e_N. For a more detailed description of the contact law and impact law see e.g. [3]. The concept of measure differential inclusions with set-valued contact and impact law can only be used for the mixed shooting-HBM and full shooting method. As discussed in Section 3.1, the HBM only allows a smoothed contact law. Therefore, the contact for the HBM is modeled using a one-sided spring-damper element

$$-\lambda_N^{smoothed} = \begin{cases} k_c g_N + d_c \gamma_N & g_N \leq 0 \\ 0 & g_N > 0. \end{cases} \tag{30}$$

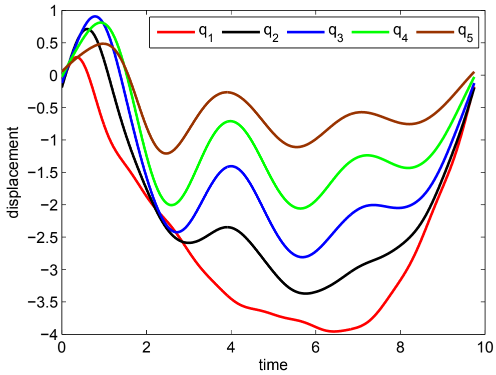
The equivalent restitution coefficient e_N for a specific one-sided spring-damper element can be calculated following Brogliato [2]. Since, only a non-dissipative, elastic contact ($e_N = 1$) is used in this work d_c is defined as zero and the model tends to the hard contact if $\lim_{k_c \to \infty}$.
In Figure 6, the displacements of a five DOF oscillator with a gap ($g_{N,0} = 0.1$) at the first mass for the HBM, mixed shooting HBM and the full shooting method are depicted. Figure 7 shows the velocity of the first mass. The used parameters are summarized in Table 2. Like before for the system with dry friction, the HBM has difficulties to approximate the jump in the velocity of the first mass at the collision time-instant ($t = 0.55$), although the contact stiffness k_c is chosen relatively high. The mixed and full shooting method show a true velocity jump whereas the HBM only gives a rough approximation of this phenomenon.

Table 2: Selected parameters for the system with impact.

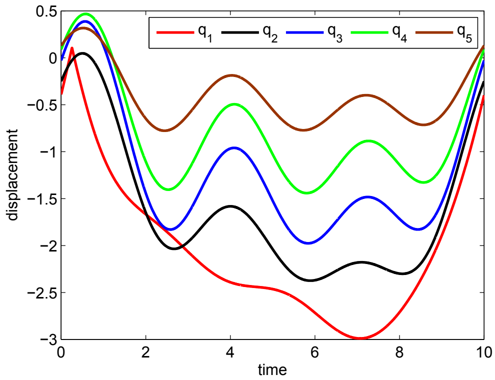
parameter	m_1	m_{2-5}	k_i	c_i	k_c	e_N	ω	$f_{ex,5}$
value	10	1	1	0.3	8000	1	$\frac{1}{5}\pi$	$5\cos(\omega t)$



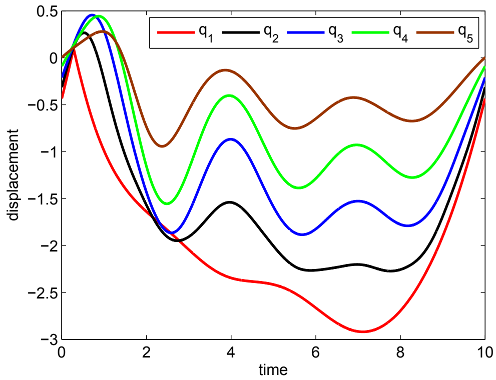
a) HBM $n_H = 3$



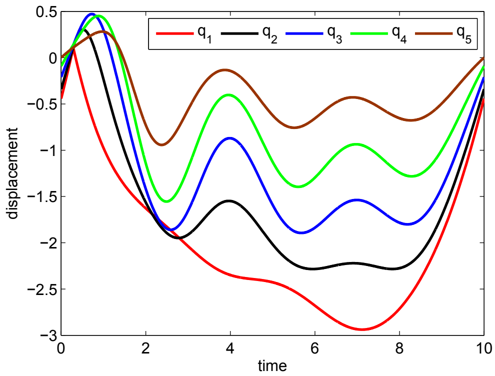
b) HBM $n_H = 9$



c) mixed $n_H = 3$



d) mixed $n_H = 9$



e) full shooting

Figure 6: Displacements of a five DOF oscillator with impact calculated with the different methods.

4 Concluding Remarks

The presented mixed shooting-HBM approach shows good characteristics in accuracy as well as in calculation effort, at least for the investigated benchmark system. Depending on the system size and the nonlinear characteristics the method can be a good alternative to the commonly used methods like HBM and shooting. It should be noted, that the numerical efficiency of the methods are hard to compare and that there exist alternative HBM methods to compute periodic solutions of systems with dry friction and impact. Further research will focus on providing a better comparison of the mixed shooting-HBM method with the existing methods.

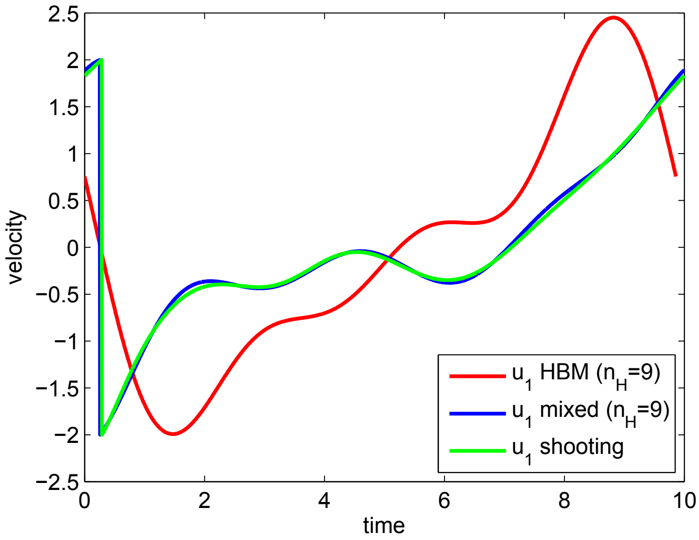


Figure 7: Velocities of the 5 DOF oscillator with impact.

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